ASYMPTOTIC SELF-SIMILARITY AND ORDER-TWO ERGODIC THEOREMS FOR RENEWAL FLOWS

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Abstract. We prove a *log average almost-sure invariance principle* (*log asip*) for renewal processes with positive i.i.d. gaps in the domain of attraction of an α–stable law with $0 < \alpha < 1$. Dynamically, this means that renewal and Mittag-Leffler paths are forward asymptotic in the scaling flow, up to a time average. This strengthens the almost-sure invariance principle in log density we proved in [FT11]. The scaling flow is a Bernoulli flow on a probability space. We study a second flow, the *increment flow*, transverse to the scaling flow, which preserves an infinite invariant measure constructed using singular cocycles. A cocycle version of the Hopf Ratio Ergodic Theorem leads to an order–two ergodic theorem for the Mittag–Leffler increment flow. Via the *log asip*, this result then passes to a second increment flow, associated to the renewal process. As corollaries, we have new proofs of theorems of [ADF92] and of [CE51], motivated by fractal geometry.

1. Introduction

Let (N_n) be a renewal process with i.i.d. positive integer gaps (X_i) of distribution function F in the domain of attraction of a completely asymmetric α–stable law with distribution function G_α, for $\alpha \in (0, 1)$. Thus, there exists a *normalizing sequence* (a_n) such that the following convergence in law holds:

$$\frac{1}{a_n}\sum_{i=1}^{n} X_i \xrightarrow{\text{law}} G_\alpha \qquad \text{as } n \to \infty. \tag{1.1}$$

For this range of α, the expected gap between events is infinite and the set $\mathcal{O}$ of event occurrences has density zero. Nevertheless, as we shall see, $\mathcal{O}$ has quite a bit of structure: it can be thought of as a fractal integer set, for which one can calculate a "dimension" and a "Hausdorff measure". Moreover, the set $\mathcal{O}$ rescales to a dilation–invariant collection of fractal subsets of the reals, which have that same dimension and average Hausdorff measure, at both small and large scales.

These observations have interesting ergodic theoretic consequences. The above renewal process describes the times of return to a subset of finite measure of a conservative ergodic infinite measure–preserving map, the *renewal transformation* (which can be represented as a countable state Markov chain with stationary infinite measure); the transformation renormalizes under scaling to an infinite measure preserving flow, the *increment flow* of the Mittag–Leffler process. This flow is transverse to the dynamics of renormalization, given by the scaling flow of index α, which due to the self–similarity of the Mittag–Leffler process preserves a probability measure. The renormalization is expressed in a

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commutation relation linking the pair of flows, identical to that shared by the geodesic and horocycle flows on Riemann surfaces of infinite area. As a consequence of the renormalization approximation, plus ergodicity of both flows, we derive an order–two ergodic theorem for the renewal flow, mirroring results for the infinite measure horocycle flow. This type of result should be considered an infinite–measure analogue of the Birkhoff ergodic theorem statement that "time average equals space average", seen through the lens of Hausdorff measure and fractal geometry [Fis92].

But first let us be more precise regarding the fractal–like structure of the set $\mathcal{O}$. One can show that for a.e. path N_n, the ratio $(\log N_n)/\log n$ converges to α, the *dimension* of the integer set $\mathcal{O}$ as defined in [BF92].

This result, noted by Chung and Erdös (in Theorem 7 of [CE51]) gives n^α as a first estimate for the growth of N_n. A better approximation is given by $\widehat{a}_n = 1/(1-F(n))$, a regularly varying sequence of index α, as $N_n/\widehat{a}_n$ converges in law to the *Mittag–Leffler* distribution of index α, which has distribution function $\mathcal{M}_\alpha(x) = (1 - G_\alpha(x^{-\frac{1}{\alpha}}))$; see Theorem 7 of [Fel49].

Writing $Y_i = 1$ when an event occurs and $Y_i = 0$ otherwise, so $N_n = \sum_{i=1}^n Y_i$, and defining the *return sequence* $\overline{a}_n = \mathbb{E}(N_n)$, the expected number of events up to time n, one has the following further result of Chung and Erdös (Theorem 6 of [CE51]):

$$\lim_{k\to\infty} \frac{1}{\log \overline{a}(k)} \sum_{n=1}^{k} \frac{Y_n}{\overline{a}_n} = 1, \text{ a.s.} \tag{1.2}$$

As noted in Proposition 1 of [ADF92], when $\overline{a}_n$ is regularly varying of index $\alpha \in (0,1)$ (which will be the case here) then (1.2) is equivalent to:

$$\lim_{k\to\infty} \frac{1}{\log k} \sum_{n=1}^{k} \frac{N_n}{\overline{a}_n}\frac{1}{n} = 1, \text{ a.s.} \tag{1.3}$$

The different normalizations $a_n, \widehat{a}_n, \overline{a}_n$ can be unified. First, as shown by Lévy, a normalizing sequence a_n for (1.1) is necessarily regularly varying of index $1/\alpha$. Now as in [FT12], one can find an especially nice normalizing sequence: we construct from F an increasing C^1 function $a(\cdot)$ with regularly varying derivative so its inverse $\widehat{a} \equiv a^{-1}$ is, at integer values, asymptotically equivalent to Feller's sequence $\widehat{a}_n$, written throughout $\widehat{a}(n) \sim \widehat{a}_n$ (i.e. their ratio goes to 1). There exists, furthermore, a constant $\mathsf{c} \in (0,1)$ such that $\overline{a}_n \sim \mathsf{c} \cdot \widehat{a}(n)$; see Corollary 1.2.

Replacing $\overline{a}_n$ by $\widehat{a}(n)$ in the above averages exchanges 1 for c as the limiting value, and equation (1.3) then admits a fractal geometric interpretation: the limit gives the "Hausdorff measure" of the integer set $\mathcal{O}$ for the gauge function $\widehat{a}(\cdot)$ (see Proposition 1.1) with the number c equalling the order–two density [BF92] of the limiting fractal sets of reals.

To explain this more precisely, we move to the more general setting of biinfinite renewal processes with real gaps. For $(X_i)_{i\in\mathbb{Z}}$ an i.i.d. sequence with distribution supported on $(0,+\infty)$, defining $(S_n)_{n\in\mathbb{Z}}$ by $S_0 = 0$,

$$S_n = \begin{cases} \sum_{i=0}^{n-1} X_i \text{ for } n > 0 \\ -\sum_{i=n}^{-1} X_i \text{ for } n < 0 \end{cases}$$

and $\overline{S}(t) = S_{[t]}$, then for $\widehat{\overline{S}}(t) = \inf\{s : \overline{S}(s) > t\}$ the generalized inverse, we call $\overline{N} = \widehat{\overline{S}} - 1$ the *two–sided renewal process* with gaps (X_i). Thus, $\overline{N}(0) = 0$ and for $t \geq 0$, $\overline{N}(t)$ is the total number of events which occur up to and including time t, excluding the initial event at time 0. For the

special case of integer gaps, then $\overline{N}(0)=0$, and

$$\overline{N}(n)=\begin{cases} \sum_{i=1}^{n} Y_i & \text{for } n>0 \\ -\ \sum_{i=n+1}^{0} Y_i & \text{for } n<0. \end{cases} \tag{1.4}$$

This agrees with the usual definition of integer gap renewal process N_n for $n\geq 0$, so we use the above equation to extend to all $n\in\mathbb{Z}$, setting $N_n\equiv\overline{N}(n)$. Both the one– and two–sided processes appear in this paper.

As in (1.1), we assume that the distribution of X_i is in the domain of attraction of G_α.

We recall the definition of the Mittag–Leffler process; see also [FT11]. Let $D=D_{\mathbb{R}^+}$ denote *Skorokhod path space*: the collection of all càdlàg (right–continuous with left limits) real–valued functions on $\mathbb{R}^+$, equipped with the Skorokhod J_1–topology. For Z a completely asymmetric stable process of index $\alpha\in(0,1)$, represented on D with Borel probability measure ν, a.e. Z is increasing and the Mittag–Leffler paths $\widehat{Z}(t)$ of parameter α are the generalized inverses of the stable paths:

$$\widehat{Z}(t)=\inf\{s:\ Z(s)>t\}. \tag{1.5}$$

Denoting by $\widehat{\nu}$ the corresponding measure for the Mittag–Leffler process, the *scaling flow* on D of index α defined by

$$(\widehat{\tau}_t\widehat{Z})(x)=\frac{\widehat{Z}(e^t x)}{e^{\alpha t}} \tag{1.6}$$

is ergodic, in fact is a Bernoulli flow of infinite entropy, see Proposition 5.4.

Our first result states:

Theorem 1.1. (A log average almost-sure invariance principle for renewal processes, $\alpha\in(0,1)$) *Let $(X_i)_{i\geq 0}$ be an i.i.d. sequence of a.s. positive random variables with common distribution function F in the domain of attraction of an α–stable law. Then there exists a C^1, increasing, regularly varying function $a(\cdot)$ of index $1/\alpha$ with regularly varying derivative, such that $a(n)$ gives a normalizing sequence of (1.1), and such that, for the regulary varying function of index one $h(\cdot)\equiv a^\alpha(\cdot)$, then $\overline{N}$ and a Mittag–Leffler process $\widehat{Z}$ of index α can be redefined to live on the same probability space so as to satisfy:*

$$\lim_{T\to\infty}\frac{1}{\log T}\int_1^T\frac{|h(\overline{N}(t))-\widehat{Z}(t)|}{t^\alpha}\frac{\mathrm{d}t}{t}=\lim_{T\to\infty}\frac{1}{\log T}\int_1^T\frac{||h\circ\overline{N}-\widehat{Z}||^\infty_{[0,t]}}{t^\alpha}\frac{\mathrm{d}t}{t}=0\quad a.s. \tag{1.7}$$

where $||f-g||^\infty_{[0,t]}=\sup_{0\leq s\leq t}|f(s)-g(s)|$.

Equivalently, in dynamical terms, for $\widehat{\tau}_t$ the scaling flow of index α, there exists a joining of the laws of the Mittag–Leffler and renewal processes $\widehat{\nu}$ and $\widehat{\mu}$, such that for a.e. pair $(\widehat{Z},\overline{N})$,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T d_1^u(\widehat{\tau}_t(\widehat{Z}),\widehat{\tau}_t(h\circ\overline{N}))\,\mathrm{d}t=0,$$

with $d_1^u(f,g)=||f-g||^\infty_{[0,1]}$; we say that $h\circ\overline{N}$ is in the Cesáro–average *d_1^u–stable manifold $W^s_{CES}(\widehat{Z})$.*

We mention the importance of finding a representative for $a(\cdot)$ which is C^1 increasing with regularly varying derivative: such a function preserves log averages, see Proposition 2.5 of [FT12], and that plays a key role in the proofs.

An important element in the proof of Theorem 1.1 is a weaker statement proved in Theorem 1.1 of [FT11], an *almost–sure invariance principle in log density, asip (log)*: that for a.e. pair $(\overline{N}, \widehat{Z})$ we have:

$$||h \circ \overline{N} - \widehat{Z}||^{\infty}_{[0,T]} = o(T^{\alpha}) \text{ (log)}, \tag{1.8}$$

where (log) means off a set $B \subseteq \mathbb{R}^+$ of logarithmic density ($\equiv \lim_{T\to\infty} \frac{1}{\log T}\int_1^T \chi_B(t)\mathrm{d}t/t$) equal to zero, for χ_B the indicator function of the set B.

Now the Birkhoff Ergodic Theorem for the scaling flow $\widehat{\tau}_t$ of the Mittag–Leffler process, together with a logarithmic change of variables, tells us that $\widehat{\nu}$–a.s.

$$\lim_{T\to\infty} \frac{1}{\log T}\int_1^T \frac{\widehat{Z}(t)}{t^{\alpha}}\frac{1}{t}\,\mathrm{d}t = \lim_{T\to\infty}\frac{1}{T}\int_0^T \varphi(\widehat{\tau}_t(\widehat{Z}))\,\mathrm{d}t = \mathbb{E}(\varphi) = \mathbb{E}(\widehat{Z}(1)), \tag{1.9}$$

where $\varphi(f) = f(1)$ defines an L^1- observable on the probability space $(D, \widehat{\nu})$.

Combining this with the following further result from [FT11]: for the inverse function $\widehat{a} \equiv a^{-1}$,

$$\left|\left|\frac{\overline{N}(e^t\cdot)}{\widehat{a}(e^t)} - \widehat{\tau}_t(\widehat{Z})\right|\right|^{\infty}_{[0,1]} = o(1) \qquad \text{a.s. (log)}, \tag{1.10}$$

which was shown to follow from (1.8), we then prove:

Corollary 1.1. *Under the assumptions and notation of Theorem 1.1, $\widehat{\mu}$–a.s.*

$$\lim_{T\to\infty}\frac{1}{\log T}\int_1^T \frac{\overline{N}(t)}{\widehat{a}(t)}\frac{\mathrm{d}t}{t} = \lim_{T\to\infty}\frac{1}{\log T}\int_1^T \frac{h(\overline{N}(t))}{t^{\alpha}}\;\frac{\mathrm{d}t}{t} = \mathbb{E}(\widehat{Z}(1)). \tag{1.11}$$

To explain the connection of the theorem and corollary with fractal geometry, we note that the paths $\widehat{Z}(t)$ are nondecreasing and continuous, with a nowhere dense set $C_{\widehat{Z}}$ of points of increase. By a result of Hawkes, for the gauge function

$$\psi(t) = t^{\alpha}(\log\log\frac{1}{t})^{1-\alpha}, \text{ then for } c_{\alpha} = \frac{\alpha^{1-\alpha}(1-\alpha)^{\alpha}}{\Gamma(3-\alpha)}, \tag{1.12}$$

$\widehat{Z}(t)/c_{\alpha}$ is the distribution function of the Hausdorff measure H_{ψ} restricted to the set $C_{\widehat{Z}}$; see §6.4. That is, a.e. Mittag–Leffler path $\widehat{Z}(t)$ is the $c_{\alpha}H_{\psi}$–Cantor function for the α–dimensional fractal set $C_{\widehat{Z}}$, and the content of Theorem 1.1 is a precise description of the convergence of the discrete set of events of the renewal process to this fractal set of reals.

An application of the Ergodic Theorem then allows us to relate the limiting constant of (1.11) to small–scale fractal geometry:

Proposition 1.1. *For ψ as in* (1.12)*, the constant* $\mathsf{c} \equiv \mathbb{E}(\widehat{Z}(1)) = \int f(1)d\widehat{\nu}(f)$ *is equal to c_{α} times the one–sided order–two density of the set of points of increase of the process $\widehat{Z}$ with respect to H_{ψ}: for $\widehat{\nu}$–a.e. $\widehat{Z}$, for H_{ψ}-a.e point $x \in C_{\widehat{Z}}$,*

$$\mathsf{c} = c_{\alpha}\cdot\lim_{T\to\infty}\frac{1}{T}\int_0^T \frac{H_{\psi}(C_{\widehat{Z}}\cap[x, e^{-s}])}{e^{-s\alpha}}\mathrm{d}s = \frac{\sin\pi\alpha}{\pi\alpha}.$$

Now we turn to the analogy with the geodesic and stable horocycle flows on the unit tangent bundle of an infinite area Riemann surface uniformized by a finitely generated Fuchsian group. These flows are ergodic, and preserve, respectively, a probability measure, the Patterson-Sullivan measure [Pat76] [Sul70], and a related infinite measure, first studied by Kenny [Ken83]; see [Fis04]. Moreover, they satisfy the commutation relation

$$g_t \circ h_s = h_{e^{-t}s} \circ g_t. \tag{1.13}$$

Here, the scaling flow $\widehat{\tau}_t$ on Mittag–Leffler paths will play the role of the geodesic flow, with the part of the horocycle flow taken by the *increment flow* $\overline{\eta}_t : f(x) \mapsto f(x+t) - f(t)$ on the two–sided Skorokhod path space $D = D_{\mathbb{R}}$; the scaling flow acts on the two–sided paths again by equation (1.6), with the pair $\widehat{\tau}_t$, $\overline{\eta}_s$ obeying the same relation as for g_t, h_s.

From the renewal process we define the *renewal flow*, also realized as an increment flow on D but with a different infinite invariant measure. The commutation relation can be thought of as stating that the geodesic flow renormalizes the horocycle flow to itself, and that $\widehat{\tau}_t$ renormalizes the Mittag–Leffler increment flow to itself. Then Theorem 1.1 says, essentially, that the renewal increment flow renormalizes to the Mittag–Leffler increment flow in the limit, by applying the scaling flow to the joined pair of paths.

To state our next results, which describe the ergodic theoretic consequences, we recall these definitions; see [Aar97]. A measure preserving flow τ_t of a possibly infinite measure space (X, μ) is *ergodic* iff any *invariant* set A has $\mu A = 0$ or $\mu A^c = 0$; the flow is *recurrent* iff almost every point returns to a subset of positive measure A for arbitrarily large times, an equivalent notion being that the flow is *conservative.*

The invariant measures for both the Mittag–Leffler and renewal increment flows are constructed via *cocycles* over the flows, see §§3.2 and 5.1. These are both conservative ergodic infinite measure flows (Propositions 5.4 and 5.5). Our conclusions are also most naturally stated and proved using cocycles. In the statement, the *integral* of a cocycle Φ over the flow $(D_{\mathbb{R}}, \overline{\nu}, \overline{\eta}_t)$ is $\mathbb{I}(\Phi) \equiv \frac{1}{t}\int_D \Phi(x,t) \mathrm{d}\overline{\nu}(x)$; this does not depend on t, and agrees with the usual notion of integral of a function in the special case that the cocycle is generated by a function (Proposition 3.1).

We prove:

Theorem 1.2. *(Order–two ergodic theorems for Mittag–Leffler and renewal increment flows)*
(i) Let $\Phi(x,t)$ be a cocycle over the increment flow $(D_{\mathbb{R}}, \overline{\nu}, \overline{\eta}_t)$ on Mittag–Leffler paths which is measurable, of local bounded variation in t, and with $\mathbb{I}(\Phi)$ finite. Then for $\overline{\nu}$–a.e. Mittag-Leffler path $\widehat{Z}$, and for c as above,

$$\lim_{T\to\infty} \frac{1}{\log T} \int_1^T \frac{\Phi(\widehat{Z}, t)}{t^\alpha} \, \frac{\mathrm{d}t}{t} = \mathsf{c}\ \mathbb{I}(\Phi). \tag{1.14}$$

(ii) Assuming the gap distributions of Theorem 1.1, let $\Phi(x,t)$ be a cocycle over the increment flow on renewal paths $(D_{\mathbb{R}}, \overline{\mu}, \overline{\eta}_t)$ which is measurable, of local bounded variation in t, and with $\mathbb{I}(\Phi)$ finite. Then for $\overline{\mu}$–a.e. renewal path $\overline{N}$, and for $\widehat{a}(\cdot)$ and c as above, we have:

$$\lim_{T\to\infty} \frac{1}{\log T} \int_1^T \frac{\Phi(\overline{N}, t)}{\widehat{a}(t)} \, \frac{\mathrm{d}t}{t} = \mathsf{c}\ \mathbb{I}(\Phi). \tag{1.15}$$

One of the steps in the proof is to show, in Proposition 6.1, that the increment flow satisfies, for both the Mittag–Leffler and renewal measures, a property akin to the defining property of the classical horocycle flow: that the stable manifolds of the geodesic flow are preserved. In our setting

the stable manifolds are weakened to the Cesàro–average stable manifolds, leading to this precise statement: for $\widehat{\nu}$–a.e. $\widehat{Z}$, for all $t \in \mathbb{R}$, $\eta_t \widehat{Z}$ is in $W^s_{CES}(\widehat{Z})$, while with respect to the joining of Theorem 1.1, for a.e. pair $(\widehat{Z}, \overline{N})$, for all $t \in \mathbb{R}$, $\eta_t(h \circ \overline{N})$ is in $W^s_{CES}(\widehat{Z})$.

Specializing next to the case of integer gaps with which we began the paper, we consider a countable-state Markov chain. The ergodic theory model is the following: we give the *alphabet* $\mathcal{A}$ (this is the collection of states) the discrete topology and the biinfinite path space $\Pi = \Pi_{-\infty}^{+\infty}\mathcal{A}$ the product topology, with $\mathcal{B}$ the Borel σ–algebra of Π. The left shift map σ acts on Π by $(\sigma(x))_i = x_{i+1}$, thus sending $x = (\dots x_{-1}.x_0x_1\dots)$ to $(\dots x_{-1}x_0.x_1\dots)$. We are given a row–stochastic matrix $P = (P_{ab})$, i.e. such that $\sum_{b\in\mathcal{A}} P_{ab} = 1$ for each $a \in \mathcal{A}$, and an invariant row vector $\widehat{\pi} = (\widehat{\pi}_a)_{a\in\mathcal{A}}$, so $\widehat{\pi} = \widehat{\pi}P$, with ρ the shift–invariant Markov measure determined by P and $\widehat{\pi}$. Then $(\Pi, \mathcal{B}, \rho, \sigma)$ is a measure–preserving transformation (of infinite measure iff $\widehat{\pi}$ has infinite mass), with $x = (x_i)_{i\in\mathbb{Z}}$ a path of the time biinfinite stationary Markov chain.

Choosing a state $\mathtt{a} \in \mathcal{A}$, let us assume that ρ is normalized so that for $A = \{x : x_0 = \mathtt{a}\}$, $\rho(A) = 1$. For $i \in \mathbb{Z}$ define $Y_i = \chi_A(\sigma^i(x))$, so $Y_i = 1$ when the event $\mathtt{a}$ occurs. Determining N_n by equation (1.4), with N_n replacing $\overline{N}(n)$, this defines the *occupation time process* of the state $\mathtt{a}$.

We show:

Corollary 1.2. *Given a stationary infinite measure conservative ergodic Markov chain $(x_i)_{i\in\mathbb{Z}}$ taking values in a countable alphabet $\mathcal{A}$, so $x = (x_i)_{i\in\mathbb{Z}} \in \Pi = \Pi_{-\infty}^{\infty}\mathcal{A}$, assume that for some state $\mathtt{a} \in \mathcal{A}$, the shift–invariant measure ρ on Π is normalized so that $\rho[x_0 = \mathtt{a}] = 1$. Then for N_n the occupation time process of $\mathtt{a}$, these conditions are equivalent:*

(a) the return sequence $\overline{a}_n \equiv \mathbb{E}(N_n)$ is regularly varying of index $\alpha \in (0, 1)$;

(b) the distribution function F of return times to state $\mathtt{a}$ is in the domain of attraction of G_α.

Suppose that (a) (or (b)) holds. Then letting σ denote the left shift map on (Π, ρ), we have that for any $\varphi \in L^1(\Pi, \rho)$, writing $S_n\varphi = \sum_{i=0}^{n-1} \varphi(\sigma^i x)$ for $n > 0$, for ρ–a.e. x,

$$(i) \qquad \lim_{k\to\infty} \frac{1}{\log k} \sum_{1}^{k} \frac{S_n\varphi(x)}{\widehat{a}(n)} \frac{1}{n} = \mathtt{c} \int_\Pi \varphi \mathrm{d}\rho, \tag{1.16}$$

$$(ii) \qquad \lim_{k\to\infty} \frac{1}{\log \widehat{a}(k)} \sum_{n=1}^{k} \frac{\varphi \circ \sigma^n(x)}{\widehat{a}(n)} = \mathtt{c} \int_\Pi \varphi \mathrm{d}\rho, \tag{1.17}$$

for $\widehat{a}(\cdot)$ and $\mathtt{c}$ as above.

(iii) Moreover, $\overline{a}_n \sim \mathtt{c} \cdot \widehat{a}(n)$ hence (1.16), (1.17) are equivalent to results of [ADF92] *for this case, with (1.17) implying a theorem of* [CE51].

In particular, the corollary applies to the *renewal transformation*, the infinite measure transformation built from a renewal process with integer gaps; one of the models for this map is a countable state Markov chain. See §5.3.

The outline of the paper is as follows. In §2 we prove Theorem 1.1. This follows from a key result, Lemma 2.3, proved in §2.2. Its proof hinges upon two preparatory lemmas regarding normalized partial sums of i.i.d. variables in the domain of attraction of the stable law G_α, stated and proved in §2.1, as well as an idea of Ibragimov and Lifshits. In §2.3 we show that Lemma 2.3 yields Theorem 1.1. Lastly, the proof of Corollary 1.1 is presented at the end of §2.3.

Then we turn to the ergodic theory: the construction of the flows and measures, and the proof of Theorem 1.2. To carry this out, in §3 we develop some suitable machinery in abstract ergodic

theory regarding cocycles and dual flows,, and prove a cocycle version of the Hopf ratio ergodic theorem. In §4 we study the scaling and increment flows and their duals. In §5 we use cocycles to construct our invariant measures. In §5.3 we describe several models for the renewal transformation and explain the connection with the renewal flow: it can be seen as the suspension flow over the transformation. In §6 we first prove a general order–two ergodic theorem, valid for self–similar processes which are dual to processes with stationary increments. We then show in §6.2 that the increment flow behaves like a horocycle flow, and in §6.3 we finish the proofs of Theorem 1.2 and Corollary 1.2. The identification of the constant c in terms of fractal geometry is given in Lemma 6.2, with Proposition 1.1 proved in §6.4.

2. Proof of the log average *asip*, Theorem 1.1

We recall from §2 of [FT12] some background and notation which will be used throughout the paper; see also [Fel71] pp 312-315 and 570. For $\alpha \in (0,1)\cup(1,2]$ and $\xi \in [-1,1]$, a random variable X has *stable law* $G_{\alpha,\xi,\kappa,\theta}$ if its characteristic function (i.e. Fourier transform) is

$$\mathbb{E}(e^{iwX}) = \exp\left(i\theta w + \kappa \cdot \frac{\Gamma(3-\alpha)}{\alpha(\alpha-1)}\,|w|^\alpha \big(\cos\frac{\pi\alpha}{2} - \frac{w}{|w|} i\xi \sin\frac{\pi\alpha}{2}\big)\right). \tag{2.1}$$

There is a unique process Z with stationary independent increments and with $G_{\alpha,\xi} \equiv G_{\alpha,\xi,1,0}$ the law for $Z(1)$; Z is self–similar of index $1/\alpha$, thus the law of $Z(t)$ is $G_{\alpha,\xi,t,0}$. Now $G_{\alpha,\xi}$ has support on $(0,+\infty)$ if and only if $\alpha \in (0,1)$ and $\xi = +1$, in which case it is a (positive) *completely asymmetric* law. The corresponding process Z is known as the *stable subordinator* of index α.

A distribution function F supported on $(0,\infty)$ belongs to the domain of attraction (see (1.1)) of $G_\alpha \equiv G_{\alpha,1}$ with $\alpha \in (0,1)$ if and only if $1-F$ is regularly varying of index $-\alpha$, that is $1-F(x) = x^{-\alpha}\, l(x)$ with $l(\cdot)$ some slowly varying function (i.e. for all $x > 0$, $l(tx)/l(t) \to 1$ as $t \to \infty$). In this paper regular variation always means at $+\infty$ unless indicated otherwise.

The proof of Thm. 1.1 is carried out in several steps. We begin with some preparatory material.

2.1. **Two lemmas.**

Lemma 2.1. *Let $(X_i)_{i\geq 0}$ be an i.i.d. sequence of a.s. positive variables with common distribution function F in the domain of attraction of $G_\alpha \equiv G_{\alpha,1}$ for $\alpha \in (0,1)$. So by* (1.1) *there exists a regularly varying sequence a_k such that S_k/a_k converges in law to $Z(1)$, with $\alpha-$stable distribution.*
(i) (de Acosta and Giné, see [DG79]*, page 225) For any $0 \leq \beta < \alpha$, we have*

$$\lim_{k\to\infty} \mathbb{E}\left(\left(\frac{S_k}{a_k}\right)^\beta\right) = \mathbb{E}(Z^\beta(1)) < \infty.$$

(ii) For any $p > 0$, we have

$$\lim_{k\to\infty} \mathbb{E}\left(\left(\frac{a_k}{S_k}\right)^p\right) = \mathbb{E}\left(\frac{1}{Z^p(1)}\right) = \mathbb{E}(\widehat{Z}^{p/\alpha}(1)) < \infty. \tag{2.2}$$

Equivalently,

$$\lim_{k\to\infty} \int_0^1 \mathbb{P}(S_k \leq t a_k)\, \frac{\mathrm{d}t}{t^{1+p}} = \int_0^1 \mathbb{P}(Z(1) \leq t)\, \frac{\mathrm{d}t}{t^{1+p}} < \infty. \tag{2.3}$$

We shall also need the following

Lemma 2.2. *Under the assumptions of Theorem 1.1, we set*

$$\xi_k(t) = \chi_{S_k \le ta(k)} - \mathbb{P}(S_k \le ta(k)), \;\; t \ge 0. \tag{2.4}$$

Let $\epsilon(\cdot)$ be a positive function which goes to 0 at infinity. Denoting by $\widehat{a}(\cdot)$ the inverse of $a(\cdot)$, set $l_k = \widehat{a}(\epsilon(k)a(k))$ and let l and k be two positive integers such that $l \le l_k < k$. Then we have

$$\mathbb{E}(\xi_k(t)\xi_l(t)) \le c\, \mathbb{P}\biggl(\frac{S_l}{a(l)} \le t\biggr) \left(\frac{a(l)}{t\varepsilon(k)a(k)}\right)^{\alpha^-} + \; \mathbb{P}\biggl(t(1-\varepsilon(k)) \le \frac{S_k}{a(k)} \le t(1+\varepsilon(k))\biggr),$$

for l large, with c some positive constant and $0 < \alpha^- < \alpha$.

We now move on to the proofs of these lemmas.

Proof of Lemma 2.1, (ii). Since F belongs to the domain of attraction of G_α, we have

$$1 - F(x) = x^{-\alpha}\, L(x), \tag{2.5}$$

for L some slowly varying function. Now by (1.1), for all $p > 0$ we have that $(a_k/S_k)^p$ converges in law to $1/Z^p(1)$. From page 32 of [Bil68], the convergence of moments (ii) will follow from the uniform integrability of $((a_k/S_k)^p)_{k \ge k_o}$ for k_0 large, and it is enough to check that for some $p' > p$

$$\sup_{k \ge k_0} \mathbb{E}\left(\left(\frac{a_k}{S_k}\right)^{p'}\right) < \infty. \tag{2.6}$$

For any positive random variable X, using Fubini-Tonelli we have $\mathbb{E}(X) = \int_0^\infty \mathbb{P}(X \ge t)\, \mathrm{d}t$, so

$$\mathbb{E}\left(\left(\frac{a_k}{S_k}\right)^{p'}\right) = p' \left(\int_0^1 + \int_1^\infty\right) \mathbb{P}(S_k \le ta_k)\, \frac{\mathrm{d}t}{t^{1+p'}}. \tag{2.7}$$

The integral over $[1, \infty)$ is bounded. We find an upper bound for that over $[0, 1]$, writing

$$\mathbb{P}(S_k \le ta_k) \le \mathbb{P}(\max_{1 \le i \le k} X_i \le ta_k) = (\mathbb{P}(X_1 \le ta_k))^k = \bigl(F(ta_k)\bigr)^k,$$

as the X_i are i.i.d. with common distribution function F. From (2.5), it follows that $\forall t > 0$ and k large enough,

$$\mathbb{P}(S_k \le ta_k) \le \exp\left(k \log\left(1 - \frac{L(ta_k)}{t^\alpha a_k^\alpha}\right)\right) \le \exp\left(-k\, \frac{L(ta_k)}{t^\alpha a_k^\alpha}\right),$$

where we have used the fact that $L(u) = o(u^\alpha)$. Now, $a_k^\alpha \sim kL(a_k)$ and by Potter's Theorem (see [BGT87], page 25) for any choice of $A > 1$ and $\delta > 0$,

$$\frac{L(y)}{L(x)} \le A \max\bigl((y/x)^\delta, (y/x)^{-\delta}\bigr), \text{ for } x, y \text{ large enough.} \tag{2.8}$$

Taking $x = ta_k$ and $y = a_k$, this gives a lower bound for $L(ta_k)/L(a_k)$. Thus,

$$\mathbb{P}(S_k \le ta_k) \le \exp\left(-\frac{c}{t^{\alpha-\delta}}\right) \quad (t \in (0, 1)), \tag{2.9}$$

for some $c > 0$, and so the integral over $[0, 1]$ in (2.7) is finite. This finishes the proof of (2.6).

From the definition of $\widehat{Z}$ (see (1.5)), the fact that Z is the generalized inverse of $\widehat{Z}$, the continuity of $\widehat{Z}$, and finally the self–similarity of Z, we have:

$$\forall t \geq 0, \;\; \mathbb{P}(\widehat{Z}(1) \leq t) = \mathbb{P}(Z(t) \geq 1) = \mathbb{P}(t^{1/\alpha}\, Z(1) \geq 1) = \mathbb{P}\left(\frac{1}{Z^{\alpha}(1)} \leq t\right). \tag{2.10}$$

Thus $\widehat{Z}(1)$ has the same law as $1/Z^{\alpha}(1)$, and as $\widehat{Z}(1)$ has finite moments of all order (see [BGT87], page 337) we are done with the proof of (2.2). Lastly, (2.9) together with Lebesgue's Dominated Convergence Theorem implies (2.3), completing the proof of (ii). □

Proof of Lemma 2.2. From the definition of $\xi_{\cdot}(t)$, we get:

$$\mathbb{E}(\xi_k(t)\xi_l(t)) = \mathbb{P}(S_l \leq ta(l); S_k \leq ta(k)) - \mathbb{P}(S_l \leq ta(l))\, \mathbb{P}(S_k \leq ta(k)),$$

with $\mathbb{P}(A\,;\,B)$ standing for $\mathbb{P}(A \cap B)$.

Since $l < k$, S_l is independent of $S_k - S_l$; moreover since the X_i are a.s. positive we have $S_l < S_k$ a.s. Thus $\mathbb{E}(\xi_k(t)\xi_l(t))$ can be rewritten as

$$\int_0^{ta(l)} \mathbb{P}(S_k - S_l \leq ta(k) - x)\, \mathrm{d}\mathbb{P}_{S_l}(x) - \mathbb{P}(S_l \leq ta(l)) \int_0^{ta(k)} \mathbb{P}(S_k - S_l \leq ta(k) - x)\, \mathrm{d}\mathbb{P}_{S_l}(x)$$

where $\mathbb{P}_{S_l}$ denotes the law of S_l or also the pushed–forward measure $S_l^*(\mathbb{P})$. Splitting the last integral $\int_0^{ta(k)}$ into $\int_0^{ta(l)} + \int_{ta(l)}^{ta(k)}$, this equals

$$\mathbb{P}(S_l > ta(l)) \int_0^{ta(l)} \mathbb{P}(S_k{-}S_l \leq ta(k){-}x)\mathrm{d}\mathbb{P}_{S_l}(x){-}\mathbb{P}(S_l \leq ta(l)) \int_{ta(l)}^{ta(k)} \mathbb{P}(S_k{-}S_l \leq ta(k){-}x)\mathrm{d}\mathbb{P}_{S_l}(x).$$

Now, since $a(l_k) = \varepsilon(k)a(k) \leq a(k)$ for k large then

$$\begin{aligned} \mathbb{E}(\xi_k(t)\xi_l(t)) \;&\leq\; \mathbb{P}(S_l > ta(l)) \int_0^{ta(l)} \mathbb{P}(S_k - S_l \leq ta(k) - x)\mathrm{d}\mathbb{P}_{S_l}(x) \\ &-\; \mathbb{P}(S_l \leq ta(l)) \int_{ta(l)}^{ta(l_k)} \mathbb{P}(S_k - S_l \leq ta(k) - x)\mathrm{d}\mathbb{P}_{S_l}(x). \end{aligned}$$

Next for $x \leq ta(l_k) = t\varepsilon(k)a(k)$ we have $\mathbb{P}(S_k - S_l \leq ta(k)(1 - \varepsilon(k))) \leq \mathbb{P}(S_k - S_l \leq ta(k) - x)$; moreover for any $x > 0$, $\mathbb{P}(S_k - S_l \leq ta(k) - x) \leq \mathbb{P}(S_k - S_l \leq ta(k))$. Thus $\mathbb{E}(\xi_k(t)\xi_l(t))$ is less than or equal to $\mathbb{P}(S_l \leq ta(l))$ times

$$\begin{aligned} &\mathbb{P}(S_l > ta(l))\, \mathbb{P}(S_k - S_l \leq ta(k)) - \mathbb{P}(ta(l) < S_l \leq t\varepsilon(k)a(k))\, \mathbb{P}(S_k - S_l \leq ta(k)(1 - \varepsilon(k))) \\ \stackrel{def}{=}\; &\mathbb{P}(S_l \leq ta(l))\, (xy - x'y') \end{aligned}$$

with $x, y, x^{'}, y^{'} \in [0,1]$ such that $x^{'} \leq x$ and $y^{'} \leq y$. Since $xy - x^{'}y^{'} \leq (x - x^{'}) + (y - y^{'})$, putting all the previous pieces together gives:

$$\begin{aligned} \mathbb{E}(\xi_k(t)\xi_l(t)) &\leq \mathbb{P}(S_l \leq ta(l))\Big(\mathbb{P}(S_l > t\varepsilon(k)a(k)) + \mathbb{P}(t(1 - \varepsilon(k))a(k) \leq S_k - S_l \leq ta(k))\Big) \\ &= \mathbb{P}(S_l \leq ta(l))\mathbb{P}(S_l > t\varepsilon(k)a(k)) + \mathbb{P}\Big(S_l \leq ta(l);\; t(1 - \varepsilon(k))a(k) \leq S_k - S_l \leq ta(k)\Big) \end{aligned}$$

as S_l is independent of $S_k - S_l$. Now since $a(l) \le a(l_k) = \varepsilon(k)a(k)$, this is

$$\begin{aligned} &\le \mathbb{P}\Big(\frac{S_l}{a(l)} \le t\Big)\, \mathbb{P}\Big(\frac{S_l}{a(l)} > t\frac{a(l_k)}{a(l)}\Big) + \mathbb{P}\Big(t(1-\varepsilon(k)) \le \frac{S_k}{a(k)} \le t(1+\varepsilon(k))\Big) \\ &\le c\,\mathbb{P}\Big(\frac{S_l}{a(l)} \le t\Big)\, \Big(\frac{a(l)}{ta(l_k)}\Big)^{\alpha^-} + \mathbb{P}\Big(t(1-\varepsilon(k)) \le \frac{S_k}{a(k)} \le t(1+\varepsilon(k))\Big), \end{aligned}$$

where we have used Markov's inequality, part (i) of Lemma 2.1 (with $\beta = \alpha^-$ and $a_k \equiv a(k)$) to guarantee that $\mathbb{E}(S_l/a(l))^{\alpha^-} \le c$ for some constant $c > 0$. This finishes the proof of Lemma 2.2. □

2.2. **The proof of (1.7).** Using the previous two lemmas, this will follow from the result we now state and prove.

Lemma 2.3. *Under the assumptions of Theorem 1.1, there exists some constant $c > 0$ such that for n_0 large enough, we have*

$$\limsup_{n\to\infty} \frac{1}{\log n} \sum_{n_0 \le k \le n} \frac{1}{k} \left(\frac{a(k)}{S_k}\right)^{\alpha p} \le c, \quad a.s. \tag{2.11}$$

We begin by proving the lemma then move on to showing how (1.7) can be derived from this.

Proof of Lemma 2.3. We write

$$P_n \equiv \Big(\sum_{n_0 \le k \le n} \frac{1}{k}\Big)^{-1} \sum_{n_0 \le k \le n} \frac{1}{k}\delta_{S_k/a(k)},$$

where δ_b is Dirac mass at b. So proving (2.11) is equivalent to showing that

$$\limsup_{n\to\infty} \int \phi \,\mathrm{d}P_n \le c, \quad a.s. \tag{2.12}$$

where

$$\phi(x) = x^{-\beta} (x > 0), \quad \beta \equiv \alpha p, \quad (\alpha < 1, p > 1).$$

From the a.s. CLT, see [BD93], page 1643, the sequence of probability measures (P_n) converges a.s. weakly to $Z(1)$. It then follows from page 31 of [Bil68] that a.s. for any *bounded* function ψ with Lebesgue–negligible set of discontinuities, $\int \psi \,\mathrm{d}P_n$ converges to $\mathbb{E}(\psi(Z(1))$, as $n \to \infty$.

This does not apply to our function ϕ as it is not bounded. We circumvent this difficulty by first expressing $\int \phi \,\mathrm{d}P_n$ as:

$$\int \phi \,\mathrm{d}P_n = \int_0^\infty P_n((0, \frac{1}{t^{1/\beta}}])\,\mathrm{d}t = \beta \left(\int_0^1 + \int_1^\infty\right) P_n((0,x])\,\frac{\mathrm{d}x}{x^{1+\beta}},$$

where we have used a Fubini argument. The integral over $[1,\infty)$ is bounded so proving (2.12) is equivalent to checking that

$$\limsup_{n\to\infty} \int_0^1 F_n(x)\,\frac{\mathrm{d}x}{x^{1+\beta}} \le c, \quad a.s.$$

where

$$F_n(x) = \frac{1}{\log n} \sum_{n_0 \le k \le n} \frac{1}{k}\,\chi_{\frac{S_k}{a(k)} \le x} \;\; \Big(= \frac{1}{\log n}\Big(\sum_{n_0 \le k \le n} \frac{1}{k}\Big)\, P_n((0,x])\Big).$$

Defining $F_\infty(x) \equiv \mathbb{P}(Z(1) \le x)$, we shall prove that a.s.:

$$\limsup_{n\to\infty} \int_0^1 F_n(x) \frac{\mathrm{d}x}{x^{1+\beta}} \le 2 \int_0^1 F_\infty(x) \frac{\mathrm{d}x}{x^{1+\beta}}. \tag{2.13}$$

For simplicity, we set

$$I_n \equiv \int_0^1 F_n(x) \frac{\mathrm{d}x}{x^{1+\beta}} \text{ and } I_\infty \equiv \int_0^1 F_\infty(x) \frac{\mathrm{d}x}{x^{1+\beta}};$$

note that by (ii) of Lemma 2.1, $I_\infty < \infty$.

Following the proof of Lemma 2 of [IL98], it is enough to show that the lim sup in (2.13) can be taken along the subsequence $n_m \equiv [e^{\gamma^m}]$, for $\gamma > 1$. Indeed, from the definition of F_n the mapping $n \mapsto \log n \times F_n$ is nondecreasing and hence so is $n \mapsto \log n \times I_n$. Thus for all $n_{m-1} \le n \le n_m$:

$$I_n \le I_{n_m} \frac{\log n_m}{\log n} \le I_{n_m} \frac{\log n_m}{\log n_{m-1}}$$

and the last ratio is asymptotically equivalent to γ.

Accordingly, assuming that a.s. $\limsup_m I_{n_m} \le 2I_\infty$, this yields:

$$\limsup_{n\to\infty} I_n \le \gamma \limsup_{m\to\infty} I_{n_m} \le 2\gamma I_\infty, \quad a.s.$$

which delivers (2.13) by letting γ decrease to one.

We now prove that $\limsup_m I_{n_m} \le 2I_\infty$, a.s. By Borel–Cantelli, this would follow from:

$$\limsup_{n\to\infty} \mathbb{E}(I_n) \le I_\infty \text{ and } \sum_m \mathrm{Var}(I_{n_m}) < \infty. \tag{2.14}$$

Beginning with $\mathbb{E}(I_n)$, we rewrite it as:

$$\mathbb{E}(I_n) = \frac{1}{\log n} \sum_{n_0 \le k \le n} \frac{1}{k} \int_0^1 \mathbb{P}(S_k \le a(k)x) \frac{\mathrm{d}x}{x^{1+\beta}}.$$

Using (2.3), the above integral converges to $\int_0^1 \mathbb{P}(Z(1) \le x)\, x^{-1-\beta}\, \mathrm{d}x$, thus a fortiori its log average does, giving that $\lim_{n\to\infty} \mathbb{E}(I_n) = I_\infty$.

We move on to considering $\mathrm{Var}(I_n)$. By definition,

$$\mathrm{Var}(I_n) = \mathbb{E}\left(\int_0^1 (F_n(t) - \mathbb{E}(F_n(t))) \frac{\mathrm{d}t}{t^{1+\beta}}\right)^2 = \mathbb{E}\left(\int_0^1 T_n(t) \frac{\mathrm{d}t}{t^{1+\beta}}\right)^2,$$

where

$$T_n(t) = \frac{1}{\log n} \sum_{n_o \le k \le n} \frac{\xi_k(t)}{k}, \tag{2.15}$$

with $\xi_k(t)$ defined as in (2.4). So for any $\varepsilon > 0$ small enough, by Jensen's inequality, we have:

$$\mathrm{Var}(I_n) = \frac{1}{\varepsilon^2} \mathbb{E}\left(\int_0^1 \frac{T_n(t)}{t^{\beta+\varepsilon}}\, \varepsilon\, \frac{\mathrm{d}t}{t^{1-\varepsilon}}\right)^2 \le \frac{1}{\varepsilon} \int_0^1 \frac{\mathbb{E}(T_n^2(t))}{t^{1+2\beta+\varepsilon}} \mathrm{d}t.$$

So as to prove that this last integral is summable along (n_m), we rewrite it as:

$$\begin{aligned}
\int_0^1 \frac{\mathbb{E}(T_n^2(t))}{t^{1+2\beta+\varepsilon}}\mathrm{d}t &= \int_0^1 \frac{1}{\log^2 n}\left(\sum_{n_0\le k,l\le n}\frac{1}{kl}\,\mathbb{E}(\xi_k(t)\xi_l(t))\right)\,\frac{\mathrm{d}t}{t^{1+2\beta+\varepsilon}}\\
= &\int_0^1 \frac{1}{\log^2 n}\left(\sum_{n_0\le k\le n}\frac{1}{k^2}\,\mathbb{E}(\xi_k^2(t))\right)\,\frac{\mathrm{d}t}{t^{1+2\beta+\varepsilon}} + 2\int_0^1 \frac{1}{\log^2 n}\left(\sum_{n_0\le l<k\le n}\frac{1}{kl}\,\mathbb{E}(\xi_k(t)\xi_l(t))\right)\,\frac{\mathrm{d}t}{t^{1+2\beta+\varepsilon}}\\
\stackrel{def}{=} &\ (\mathrm{I}) + 2\ (\mathrm{II}).
\end{aligned}$$

Let us begin with (I). By definition of $\xi_k(t)$, we have that $\mathbb{E}(\xi_k(t)^2)\le \mathbb{P}(S_k\le ta(k))$. Accordingly,

$$(\mathrm{I}) \le \frac{1}{\log^2 n}\sum_{n_0\le k\le n}\frac{1}{k^2}\int_0^1 \mathbb{P}(S_k\le ta(k))\frac{\mathrm{d}t}{t^{1+2\beta+\varepsilon}}.$$

From (2.3), the above integral is bounded hence so is the last sum, and therefore

$$(\mathrm{I}) = O\left(\frac{1}{\log^2 n}\right),$$

which is summable along (n_m) (recall that $n_m = [e^{\gamma^m}]$ with $\gamma > 1$).

We next show that the same holds true for (II).

Setting $l_k \equiv \widehat{a}(\varepsilon(k)a(k))$ with $\varepsilon(t)\equiv \frac{1}{(\log\log t)^2}$ and $\widehat{a}(\cdot)$ the inverse of $a(.)$, the sum involved in (II) splits into two parts according as to whether $l\le l_k$ or not; thus,

$$(\mathrm{II}) = \frac{1}{\log^2 n}\left(\sum_{n_0\le k\le n}\ \sum_{n_0\le l\le l_k} + \sum_{n_0\le k\le n}\ \sum_{l_k<l<k}\right)\frac{1}{kl}\int_0^1 \mathbb{E}(\xi_k(t)\xi_l(t))\frac{\mathrm{d}t}{t^{1+2\beta+\varepsilon}} \stackrel{def}{=} (\mathrm{III}) + (\mathrm{IV}).$$

We begin by estimating the covariance $\mathbb{E}(\xi_k(t)\xi_l(t))$ and start with the case where $l_k<l<k$. Since $\mathbb{E}(\xi_k^2(t))\le \mathbb{P}(S_k\le ta(k))$, the Cauchy–Schwarz inequality yields

$$\mathbb{E}(\xi_k(t)\xi_l(t)) \le \sqrt{\mathbb{E}(\xi_k(t)^2)\mathbb{E}(\xi_l(t)^2)} \le \sqrt{\mathbb{P}(S_k\le ta(k))}.$$

Hence

$$\begin{aligned}
(\mathrm{IV}) \quad &\le\ \frac{1}{\log^2 n}\sum_{n_0\le k\le n}\frac{1}{k}\left(\int_0^1 \sqrt{\mathbb{P}(S_k\le ta(k))}\frac{\mathrm{d}t}{t^{1+2\beta+\varepsilon}}\right)\sum_{l_k<l<k}\frac{1}{l}\\
&\le\ \frac{2}{\log^2 n}\sum_{n_0\le k\le n}\frac{1}{k}\ \log\frac{k}{[l_k]}\ \left(\int_0^1\sqrt{\frac{\mathbb{P}(S_k\le ta(k))}{t^{1+4\beta+2\varepsilon}}}\ \frac{1}{2}\ \frac{\mathrm{d}t}{\sqrt{t}}\right)\\
&\le\ \frac{\sqrt{2}}{\log^2 n}\sum_{n_0\le k\le n}\frac{1}{k}\ \log\frac{\widehat{a}(a(k))}{[\widehat{a}(\varepsilon(k)a(k))]}\sqrt{\int_0^1 \frac{\mathbb{P}(S_k\le ta(k))}{t^{1+4\beta+2\varepsilon}}\ \frac{\mathrm{d}t}{\sqrt{t}}}
\end{aligned}$$

by Jensen's inequality. Now using (2.3), the integral lying under the square root above stays bounded. Next, since $\widehat{a}(\cdot)$ is regularly varying of index α, Potter's Theorem (2.8) states that

picking arbitrary $A > 1$ and $0 < \rho < \alpha$, for $x \le y$ large enough, we have $\widehat{a}(y)/\widehat{a}(x) \le A\,(y/x)^{\alpha+\rho}$. Using this together with an easy computation, by our choice of $\varepsilon(\cdot)$, we arrive at:

$$\begin{aligned}
(\mathrm{IV}) &= O\left(\frac{1}{\log^2 n}\sum_{n_0\le k\le n}\frac{1}{k}\,\log\frac{\widehat{a}(a(k))}{\widehat{a}(\varepsilon(k)a(k))}\right) + O\left(\frac{1}{\log^2 n}\right)\\
&= O\left(\frac{1}{\log^2 n}\sum_{n_0\le k\le n}\frac{1}{k}\log\log\log k\right) + O(\frac{1}{\log n}) = O\left(\frac{\log\log\log n}{\log n}\right),
\end{aligned}$$

which is summable along (n_m).

Now we turn to (III). Using the estimate of $\mathbb{E}(\xi_k(t)\xi_l(t))$ for $l \le l_k$ given in Lemma 2.2, we write

$$\begin{aligned}
(\mathrm{III}) \le \frac{c}{\log^2 n}&\sum_{n_0\le k\le n}\frac{1}{k(\varepsilon(k)a(k))^{\alpha^-}}\sum_{n_0\le l\le l_k}\frac{a(l)^{\alpha^-}}{l}\int_0^1\mathbb{P}\left(\frac{S_l}{a(l)}\le t\right)\frac{\mathrm{d}t}{t^{1+2\beta+\varepsilon+\alpha^-}}\\
+\ &\frac{1}{\log^2 n}\sum_{n_0\le k\le n}\frac{1}{k}\int_0^1\mathbb{P}\Big(t(1-\varepsilon(k))\le\frac{S_k}{a(k)}\le t(1+\varepsilon(k))\Big)\frac{\mathrm{d}t}{t^{1+2\beta+\varepsilon}}\sum_{n_0\le l\le l_k}\frac{1}{l}\\
\stackrel{def}{=}\ &(\mathrm{V}) + (\mathrm{VI}).
\end{aligned}$$

We start off with (V). Since $a(\cdot)$ is regularly varying of index $1/\alpha$ then so is $a(l)^{\alpha^-}/l$ with index $\frac{\alpha^-}{\alpha} - 1 > -1$. By Karamata's Theorem ([BGT87], page 26), n_0 being large, we have:

$$\sum_{n_0\le l\le l_k}\frac{a(l)^{\alpha^-}}{l} \sim \frac{\alpha}{\alpha^-}a(l_k)^{\alpha^-} = \frac{\alpha}{\alpha^-}(a(k)\varepsilon(k))^{\alpha^-}.$$

This together with (2.3) yields $(\mathrm{V}) = O\left(\frac{1}{\log n}\right)$, which is summable along (n_m).

As for (VI), we write

$$\begin{aligned}
\textstyle\int_0^1\ &\mathbb{P}\Big(t(1-\varepsilon(k))\le\frac{S_k}{a(k)}\le t(1+\varepsilon(k))\Big)\frac{\mathrm{d}t}{t^{1+2\beta+\varepsilon}}\\
\le\ &\mathbb{E}\left(\int_{\frac{S_k}{(1+\varepsilon(k))a(k)}}^{\frac{S_k}{(1-\varepsilon(k))a(k)}}\frac{\mathrm{d}t}{t^{1+2\beta+\varepsilon}}\right) = O\left(\varepsilon(k)\,\mathbb{E}\left(\left(\frac{a(k)}{S_k}\right)^{2\beta+\varepsilon}\right)\right) = O(\varepsilon(k)),
\end{aligned}$$

by (2.2). Now, since $a(\cdot)$ is regularly varying of index $1/\alpha$ and $\varepsilon(\cdot)$ is slowly varying, we have that $\log l_k = \log\widehat{a}(\varepsilon(k)a(k)) \sim \alpha\log(\varepsilon(k)a(k)) \sim \log k$. It follows, by the choice of $\varepsilon(\cdot)$ that:

$$(\mathrm{VI}) = O\left(\frac{1}{\log^2 n}\sum_{n_0\le k\le n}\frac{\varepsilon(k)}{k}\log l_k\right) = O\left(\frac{1}{(\log\log n)^2}\right),$$

which is summable along (n_m).

We have finished the proof that $\mathrm{Var}(I_n)$ is summable along (n_m); this guarantees (2.14), hence (2.13) and then finally (2.12). The proof of Lemma (2.3) is now complete. □

2.3. **Proving that** (1.7) **follows from Lemma 2.3.** From (1.8), we have a fortiori that $|h(\overline{N}(t)) - \widehat{Z}(t)| = o(t^\alpha)$ a.s. (log). Hence, a.s. there exists a set of times $\mathcal{B}$ of log density zero such that for all $t \notin \mathcal{B}$, $|h(\overline{N}(t)) - \widehat{Z}(t)| = o(t^\alpha)$. As a result,

$$\lim_{T\to\infty} \frac{1}{\log T} \int_1^T \frac{|h(\overline{N}(t)) - \widehat{Z}(t)|}{t^\alpha} \chi_{t\notin\mathcal{B}} \frac{\mathrm{d}t}{t} = 0, \quad a.s.$$

where χ_A is the indicator function of the set A.

So proving that the first limit in (1.7) is a.s. zero is equivalent to showing that almost surely,

$$\frac{1}{\log T} \int_1^T \frac{|h(\overline{N}(t)) - \widehat{Z}(t)|}{t^\alpha} \chi_{t\in\mathcal{B}} \frac{\mathrm{d}t}{t} \longrightarrow 0, \qquad T \to \infty. \tag{2.16}$$

From Hölder's inequality, for any $p, q > 1$ such that $1/p + 1/q = 1$, the above is

$$\leq \left\{ \frac{1}{\log T} \int_1^T \left(\frac{|h(\overline{N}(t)) - \widehat{Z}(t)|}{t^\alpha} \right)^p \frac{\mathrm{d}t}{t} \right\}^{1/p} \times \left\{ \frac{1}{\log T} \int_1^T \chi_{t\in\mathcal{B}} \frac{\mathrm{d}t}{t} \right\}^{1/q}. \tag{2.17}$$

As $T \to \infty$, the last term approaches the log density of $\mathcal{B}$ to the power $1/q$, which is zero. So proving that the first term in (2.17) is eventually bounded would yield (2.16).

Now, since $p > 1$, the mapping $x \mapsto |x|^p$ is convex, and so for all $x, y \geq 0$ we have that $|x-y|^p \leq 2^{p-1}(x^p + y^p)$. Applying this for $x = h(\overline{N}(t))$ and $y = \widehat{Z}(t)$, we are done so long as the log averages of $(\widehat{Z}(t)/t^\alpha)^p$ and $(h(\overline{N}(t))/t^\alpha)^p$ are bounded.

Since the scaling flow $\widehat{\tau}_t$ for $\widehat{Z}$ is ergodic, using Birkhoff's ergodic theorem, a.s.

$$\lim_{T\to\infty} \frac{1}{\log T} \int_1^T \left(\frac{\widehat{Z}(t)}{t^\alpha} \right)^p \frac{\mathrm{d}t}{t} = \lim_{T\to\infty} \frac{1}{T} \int_0^T (\widehat{\tau}_t(\widehat{Z})(1))^p \mathrm{d}t = \mathbb{E}(\widehat{Z}(1)^p) < \infty.$$

So it remains to prove that for some positive and finite constant c we have:

$$\limsup_{T\to\infty} \frac{1}{\log T} \int_1^T \left(\frac{h(\overline{N}(t))}{t^\alpha} \right)^p \frac{\mathrm{d}t}{t} \leq c, \quad a.s. \tag{2.18}$$

The idea is to decompose the above integral on the partition $([S_n, S_{n+1}[), n \geq 0$. Since $\overline{N}$ is a renewal process, for any $S_n \leq t < S_{n+1}$ we have that $\overline{N}(t) = n$. Thus, it is enough to prove that for n_0 large enough,

$$\limsup_{T\to\infty} \frac{1}{\log T} \sum_{n_0 \leq n \leq [N(T)]} h^p(n) \int_{S_n}^{S_{n+1}} \frac{\mathrm{d}t}{t^{1+p\alpha}} \leq c \qquad a.s. \tag{2.19}$$

For this, writing S for the polygonal interpolation of S_n and $N \equiv S^{-1}$, we claim that for all A large enough, a.s. $N(T) \leq AT$, for T large. Indeed, we can write

$$\mathbb{P}(N(T) > AT) = \mathbb{P}(S(AT) < T) \leq \mathbb{P}(S_{[AT]} < T) \leq e^T \left(\mathbb{E}(e^{-X_1}) \right)^{[AT]}$$

using Markov's inequality. As $X_1 > 0$ a.s., the Laplace transform $\mathbb{E}(e^{-X_1})$ is < 1, so $\mathbb{P}(N(T) > AT)$ is exponentially small for A large enough and we are done by the Borel–Cantelli Lemma. Thus we

are reduced to proving (2.19) with $[AT]$ replacing $[N(T)]$. But

$$\limsup_{T\to\infty} \frac{1}{\log T} \sum_{n_0\le n\le [AT]} h^p(n) \int_{S_n}^{S_{n+1}} \frac{\mathrm{d}t}{t^{1+p\alpha}} \tag{2.20}$$

$$\le \limsup_{T\to\infty} \Bigg(\frac{1}{\alpha p \log T} \left(\sum_{n_0\le n\le [AT]} \frac{h^p(n)-h^p(n-1)}{S_n^{\alpha p}} \right) + \frac{1}{\alpha p \log T} \frac{h^p(n_0)}{S_{n_0}^{\alpha p}} \Bigg). \tag{2.21}$$

The last term of (2.21) a.s. converges to 0 as $T\to\infty$. And, recalling that $h(\cdot)=a^\alpha(\cdot)$ is regularly varying of index 1 and with regularly varying derivative, Karamata's theorem (see [BGT87] page 12-28) yields $sh'(s)\sim h(s)$, so for x large we have:

$$h^p(x+1)-h^p(x) = \int_x^{x+1} (h^p)'(s)\mathrm{d}s = \int_x^{x+1} p\, \frac{h^p(s)}{s} \frac{sh'(s)}{h(s)} \mathrm{d}s \sim p\, \frac{h^p(x)}{x}.$$

Thus the lim sup in (2.20) stays a.s. bounded by Lemma 2.3, and we have just showed that Lemma 2.3 implies (2.18), which in turn delivers (2.16) and thus the first equality in (1.7).

We note that since $h(\cdot)$, $\overline{N}$ and $\widehat{Z}$ are nondecreasing, it is straightforward to check (following the same pattern as before) that (2.16) holds true with $||h\circ\overline{N}-\widehat{Z}||_{[0,t]}$ replacing $|h(\overline{N}(t))-\widehat{Z}(t)|$. This shows the second identity in (1.7). The proof of Theorem 1.1 is now complete.

We move on to showing how Corollary 1.1 follows from Theorem 1.1.

Proof of Corollary 1.1. As noted in the introduction, since $\widehat{\tau}_t$ is ergodic for $\widehat{Z}$, Birkhoff's ergodic theorem gives (1.9). Thus, Theorem 1.1 implies that a.s.

$$\lim_{T\to\infty} \frac{1}{\log T} \int_1^T \frac{h(\overline{N}(t))}{t^\alpha} \frac{\mathrm{d}t}{t} = \mathbb{E}(\widehat{Z}(1)).$$

This is not quite the statement of Corollary 1.1. For this we need to run the reasoning we used in the proof of Theorem 1.1, starting with (1.10) instead of (1.8):

$$\left| \frac{\overline{N}(t)}{\widehat{a}(t)} - \frac{\widehat{Z}(t)}{t^\alpha} \right| \to 0 \quad a.s.\ (\log);$$

then, just as we proved that (2.16) followed from (2.18), all we have to check is that $\forall p>1$

$$\limsup_{T\to\infty} \frac{1}{\log T} \int_1^T \left(\frac{\overline{N}(t)}{\widehat{a}(t)}\right)^p \frac{\mathrm{d}t}{t} \le c, \quad a.s. \tag{2.22}$$

for some positive constant c. As $\widehat{a}(\cdot)$ is regularly varying of index α, by Potter's Theorem (2.8),

$$\left(\frac{\overline{N}(t)}{\widehat{a}(t)}\right)^p \le A \max\left(\left(\frac{h(\overline{N}(t))}{t^\alpha}\right)^{p_+}, \left(\frac{h(\overline{N}(t))}{t^\alpha}\right)^{p_-} \right) \le A \left(\frac{h(\overline{N}(t))}{t^\alpha}\right)^{p_+} + A \left(\frac{h(\overline{N}(t))}{t^\alpha}\right)^{p_-},$$

for $A>1$, $\delta>0$ small enough, t large enough, $p_+=p(1+\delta)$ and $p_-=p(1-\delta)$.

Choosing δ small enough so that $p_->1$ then applying (2.18) for $p=p_-$ and $p=p_+$ yields (2.22). As a result, $\overline{N}(t)/\widehat{a}(t)$ and $\widehat{Z}$ share the same log average, finishing the proof of Corollary 1.1. □

3. Cocycles, special flows and the ergodic theorem

3.1. **Special flows.** First we recall the ergodic theory construction of a special flow. Given an invertible measure–preserving transformation T of a measure space $(B, \mathcal{A}, \mu_B)$ (referred to as $(B, \mathcal{A}, \mu_B, T)$ or more simply as (B, μ_B, T)) and a measurable function $r : B \to (0, +\infty)$, we form the space

$$X \equiv \{(x, t) : x \in B,\, 0 \leq t \leq r(x)\} \tag{3.1}$$

and make the identification $(x, r(x)) \sim (Tx, 0)$. The flow τ_t is defined by $\tau_t(x, s) = (x, s + t)$; a point $(x, 0)$ in the *base* $B = \{(x, 0) : x \in B\}$ moves upwards at unit speed, flowing until it reaches the top at $(x, r(x))$, when it jumps back via the identification to $(Tx, 0)$ in the base and continues.

We write μ for the measure induced from the product of μ_B on the base with Lebesgue measure on $\mathbb{R}$. This measure is preserved by the flow; one calls (X, μ, τ_t) the *special flow* built *over* the *base map* (B, μ_B, T) and *under* the *return–time function* r. The special flow is conservative ergodic if and only if the base map is. If either the base or the flow space has finite measure, conservativity holds (by the Poincaré recurrence theorem). In the special case that μ_B is a probability measure, the measure of the flow space is equal to the expected return time:

$$\mu(X) = \mathbb{E}(r) = \int_B r(x)\mathrm{d}\mu_B(x). \tag{3.2}$$

A basic result is the converse to the construction of a special flow, known as the Ambrose–Kakutani Theorem. Given a measure–preserving flow $(X, \mathcal{A}, \mu, \tau_t)$, a *cross–section* of the flow is a subset B of X such that the orbit of a.e. point meets B for a nonempty discrete set of times (here *discrete* means a subset of $\mathbb{R}$ with no accumulation points). The σ-algebra of measurable sets $\mathcal{A}_B$ of the cross–section consists by definition of the subsets A such that the *rectangle* $A_{[a,b]} \equiv \{\tau_t(x) : x \in A, t \in [a, b]\}$ is $\mathcal{A}$–measurable for $a < b \in \mathbb{R}$; B is said to be a *measurable* cross–section if the collection of such rectangles generates $\mathcal{A}$. The *cross–section measure* μ_B is defined by $\mu_B(A) = \lim_{r\to 0} \frac{1}{r}\mu(A_{[0,r]})$.

Theorem 3.1. (Ambrose-Kakutani, [AK42]) *Let $(X, \mathcal{A}, \mu, \tau_t)$ be a conservative ergodic flow. Then there exists a measurable cross–section $B \subseteq X$ with finite measure μ_B. The flow is isomorphic to the special flow over B with return–time function $r(x) = \min\{t > 0 : \tau_t(x) \in B\}$ and with first return map $(B, \mathcal{A}_B, \mu_B, T)$ where $T(x) = \tau_{r(x)}(x)$.* □

3.2. **Cocycles.** For several different purposes below we shall need the abstract ergodic theory idea of a cocycle over a transformation or flow; we recall the definitions.

Definition 3.1. *A real–valued* cocycle *over a transformation $(X, \mathcal{A}, \mu, T)$ is a measurable function $\Psi : X \times \mathbb{Z} \to \mathbb{R}$ satisfying for a.e. x*

$$\Psi(x, n + m) = \Psi(x, n) + \Psi(T^n x, m)$$

for all $n, m \in \mathbb{Z}$. A cocycle Ψ over a flow $(X, \mathcal{A}, \mu, \tau_t)$ is a measurable function $\Psi : X \times \mathbb{R} \to \mathbb{R}$ which satisfies:

$$\Psi(x, t + s) = \Psi(x, s) + \Psi(\tau_s x, t) \tag{3.3}$$

for all $t, s \in \mathbb{R}$. (So in particular, $\Psi(x, 0) = 0$, for a.e. $x \in X$).

A measurable function $\psi : X \to \mathbb{R}$ *generates* a cocycle Ψ by summing along the orbits, defining $\Psi(x,0) = 0$ and

$$\Psi(x,n) = \begin{cases} \sum_{i=0}^{n-1} \psi(T^i x) & \text{for } n > 0 \\ -\sum_{i=n}^{-1} \psi(T^i x) & \text{for } n < 0. \end{cases} \tag{3.4}$$

For flows the sign is automatically handled by the calculus notation: we define

$$\Psi(x,t) = \int_0^t \psi(\tau_s(x))\mathrm{d}s. \tag{3.5}$$

Given a measurable cocycle over a transformation, there is a (unique) function which generates it: $\psi(x) \equiv \Psi(x,1)$. That this can fail for flows is illustrated by:

Example 1. Consider a flow (X, μ, τ_t) with cross–section $B \subseteq X$, and count the number of returns of a point x to the cross–section, by

$$N_B(x,t) = \begin{cases} \#\{s \in (0,t] : \tau_s(x) \in B\} & \text{for } t \geq 0 \\ -\#\{s \in (-t,0] : \tau_s(x) \in B\} & \text{for } t < 0 \end{cases} \tag{3.6}$$

(so $\forall x$, $N_B(x,0) = 0$.) Since this cocycle is discontinuous in t, it cannot be generated by a function.

3.3. **Integral of a cocycle.**

Definition 3.2. *The* integral *of a cocycle Ψ over a measure–preserving flow (X, μ, τ_t) is the extended-real number*

$$\mathbb{I}(\Psi) \equiv \frac{1}{t} \int_X \Psi(x,t)\mathrm{d}\mu(x) \tag{3.7}$$

for $t \in \mathbb{R} \setminus \{0\}$ (when that is defined, i.e. when it is finite, $+\infty$ or $-\infty$); if this is finite we say the cocycle is integrable. *We say a cocycle Ψ over a measure–preserving flow (X, μ, τ_t) is* nonnegative *a.s. if for a.e. x, $\Psi(x,t) \geq 0$ for all $t \geq 0$.*

Proposition 3.1. *Let (X, μ, τ_t) be a measure–preserving flow.*
(i) The integral of a cocycle (3.7) is indeed independent of t.
(ii) If the cocycle Ψ is generated by a function ψ, then $\mathbb{I}(\Psi) = \int_X \psi(x)\mathrm{d}\mu(x)$.
(iii) Let B be a flow cross–section with return map (B, μ_B, T) and return–time function r. Let Ψ be a measurable cocycle over the flow, which is either nonnegative or integrable. Then

$$\mathbb{I}(\Psi) = \int_B \Psi(x, r(x))\mathrm{d}\mu_B(x). \tag{3.8}$$

(iv) If a flow (X, μ, τ_t) is conservative ergodic, and Ψ is a cocycle with $\mathbb{I}(\Psi) > 0$, then for a.e. x, $\Psi(x,t) \to \pm\infty$ as $t \to \pm\infty$.

Proof of Proposition of 3.1. We start with (i). Writing $\lambda(t) = \int_X \Psi(x,t)\mathrm{d}\mu(x)$, since Ψ is measurable, for a.e. x, λ is a measurable function on $\mathbb{R}$. We suppose first that for all t, $\lambda(t)$ is finite. Now by the cocycle property together with the fact that μ is preserved by the flow,

$$\lambda(s+t) = \int_X \Psi(x, s+t)\mathrm{d}\mu(x) = \int_X \Psi(x,t) + \Psi(\tau_t(x), s)\mathrm{d}\mu(x) = \lambda(t) + \lambda(s),$$

so $\lambda(\cdot)$ is a measurable homomorphism on the additive group of the reals.

Thus for any $q \in \mathbb{Q}$, $\lambda(qt) = q\lambda(t)$. We show that λ is in fact $\mathbb{R}$–linear. This type of result goes back at least to Banach (Theorem 4, Chapter 1 of [Ban32]) in a more general context using Baire measurability; it can be viewed as a rigidity theorem. We give a simple direct proof for this case.

Defining $s(t) = \lambda(t)/t$, then $s(qt) = s(t)$ for all $t \in \mathbb{R}, q \in \mathbb{Q}$. Setting for $a < b$

$$A_{(a,b)} = \{t : s(t) \in (a, b)\},$$

then there exist a, b such that this has Lebesgue measure $m(A_{(a,b)}) > 0$. By the Lebesgue density theorem, m–a.e. point t of this set is a point of density. Now for such a t, the same is true for qt where $q \neq 0$, since Lebesgue measure under dilation by q is multiplied by q. We claim that for any $A_{(c,d)}$ with $b < c$, then $m(A_{(c,d)}) = 0$: if not, taking t_0 (resp. t_1) to be a point of density for $A_{(a,b)}$ (resp. $A_{(c,d)}$), then for any $\varepsilon > 0$ there is q such that qt_0 is ε-close to t_1, giving a contradiction. Taking smaller subintervals, we have that $s(t)$ is m–a.s. constant. Thus λ is a.s. linear over $\mathbb{R}$.

Let $G \subseteq \mathbb{R}$ denote this set of points t where $s(t) = a_0$ and $m(\mathbb{R} \setminus G) = 0$. But in fact $G = \mathbb{R}$; suppose that there is some w with $s(w) = b_0 \neq a_0$. Then for all $x = g + w$ with $g \in G$, $\lambda(x) = \lambda(g) + \lambda(w) = a_0 g + b_0 w$ which equals $a_0(g + w)$ iff $b_0 = a_0$. Hence for all $x \in G + w$, $s(x) \neq a_0$, but this translate of G is also a set of full Lebesgue measure, giving a contradiction.

Now consider the case where for some $t \neq 0$, $\lambda(t)$ is not finite. Define $\widetilde{\lambda}(t) = \lambda(t)$ if this is finite, $= 0$ otherwise. Then $\widetilde{\lambda}$ is a measurable, finite–valued function and is a homomorphism, so by the previous argument is linear, hence is identically zero. Thus $\lambda(t) = +\infty$ or $-\infty$ for each $t \neq 0$. Considering $t > 0$, we define $\widehat{\lambda}(t) = t$ if $\lambda(t) = +\infty$, $\widehat{\lambda}(t) = -t$ if $\lambda(t) = -\infty$. This is still a measurable homomorphism, so as before is linear, so only one choice is possible.

We have shown that if $\forall t \neq 0$ $\lambda(t)$ exists as an extended real number, then $\mathbb{I}(\Psi)$ is well–defined.

Proof of (ii): Using (i), then Fubini's theorem and the fact that μ is preserved by τ_t we have

$$\mathbb{I}(\Psi) = \frac{1}{t}\int_X \Psi(x,t)\mathrm{d}\mu(x) = \frac{1}{t}\int_X \int_0^t \psi(\tau_s(x))\mathrm{d}s\,\mathrm{d}\mu(x) = \frac{1}{t}\int_0^t \int_X \psi(\tau_s(x))\mathrm{d}\mu(x)\,\mathrm{d}s = \int_X \psi \mathrm{d}\mu.$$

Proof of (iii): By (i), for all $a > 0$,

$$\mathbb{I}(\Psi) \equiv \frac{1}{a}\int_X \Psi(x,a)\,\mathrm{d}\mu(x).$$

We first claim that if Ψ is either nonnegative or integrable, then

$$\frac{1}{a}\int_X \Psi(x,a)\mathrm{d}\mu(x) = \frac{1}{a}\int_B \left(\int_0^{r(x)} \Psi(\tau_t x, a)\mathrm{d}t\right)\mathrm{d}\mu_B(x). \tag{3.9}$$

This is just a Fubini argument, since locally $\mathrm{d}\mu = \mathrm{d}\mu_B \times \mathrm{d}t$; precisely, consider the measure space $B \times [0, +\infty)$ with product measure $\mu_B \times m$ where m is Lebesgue measure, and let X be as in (3.1); we define a function $\widehat{\Psi}$ by $\widehat{\Psi} = \Psi$ on $X \subseteq B \times [0, +\infty)$, zero elsewhere; then we apply Tonelli's or Fubini's theorem ([Roy68] pp. 269-270) depending on whether Ψ is nonnegative or integrable, to $\widehat{\Psi}$.

Now by the cocycle property, $\Psi(\tau_t x, a) = \Psi(x, t + a) - \Psi(x, t)$ so

$$\int_0^{r(x)} \Psi(\tau_t x, a)\mathrm{d}t = \int_{r(x)}^{r(x)+a} \Psi(x,t)\mathrm{d}t - \int_0^a \Psi(x,t)\mathrm{d}t.$$

Since for $x \in B$, $Tx = \tau_{r(x)}(x)$, we have again by the cocycle property, for any $s \geq 0$,

$$\Psi(x, r(x) + s) = \Psi(x, r(x)) + \Psi(Tx, s).$$

Therefore setting $s = t - r(x)$,

$$\int_{r(x)}^{r(x)+a} \Psi(x,t)\mathrm{d}t = \int_0^a \Psi(x, r(x)+s)\mathrm{d}s = \int_0^a \Psi(x, r(x)) + \Psi(Tx, s)\mathrm{d}s = a\Psi(x, r(x)) + \int_0^a \Psi(Tx, s)\mathrm{d}s.$$

So
$$\frac{1}{a}\int_0^{r(x)}\Psi(\tau_t x,a)\mathrm{d}t=\Psi(x,r(x))+\frac{1}{a}\int_0^a\Psi(Tx,s)\mathrm{d}s-\frac{1}{a}\int_0^a\Psi(x,t)\mathrm{d}t.$$
Inserting the righthand side into (3.9), since T preserves the measure μ_B, $\int_B\Psi(Tx,s)\mathrm{d}\mu_B(x)=\int_B\Psi(x,s)\mathrm{d}\mu_B(x)$ and we have (3.8).

To prove (iv), by Theorem 3.1 there exists a cross–section so we can apply part (iii). If $\mathbb{I}(\Psi)>0$ then using (iii) there exists $c>0$ and $A\subseteq B$ of positive measure, such that $\Psi(x,r(x))>c$ for all $x\in A$. Since the flow is recurrent, a.e. x enters A infinitely often. By the cocycle property plus the fact that Ψ is nondecreasing, therefore, $\Psi(x,t)\to+\infty$ as $t\to+\infty$. □

We remark that for integrable, nondecreasing Φ, part (iii) can be proved by an approximation argument, taking in (3.7) a limit as $a\to 0$; the exact "wraparound" proof just presented handles this more general case, where oscillations and $\mathbb{I}(\Psi)=+\infty$ are allowed.

3.4. **Ergodic theorems for cocycles over flows.** Here we prove cocycle versions of the Birkhoff and Hopf ergodic theorems; these results will be applied in the proof of Theorem 1.2. For transformations, the cocycle theorems are a direct consequence of the usual ergodic theorems as every cocycle is generated by a function. For flows, however, we shall need a different argument.

Let (X,μ,τ_t) be a conservative, ergodic, measure–preserving flow. The standard formulation of the a.s. ergodic theorems state: for $\psi:X\to\mathbb{R}$ in $L^1(X,\mu)$, for μ–a.e. $x\in X$,
(1) (Birkhoff Ergodic Theorem) if $\mu(X)=1$:
$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\psi(\tau_t x)\ \mathrm{d}t=\mathbb{E}(\psi)\equiv\int_X\psi\mathrm{d}\mu. \tag{3.10}$$
(2) (Hopf Ratio Ergodic Theorem) For $\mu(X)$ finite or infinite, $\psi,\varphi\in L^1(X,\mu)$ with $\int_X\psi\mathrm{d}\mu\neq 0$, then:
$$\lim_{T\to\infty}\frac{\int_0^T\varphi(\tau_t x)\ \mathrm{d}t}{\int_0^T\psi(\tau_t x)\ \mathrm{d}t}=\frac{\int_X\varphi\mathrm{d}\mu}{\int_X\psi\mathrm{d}\mu}. \tag{3.11}$$

Remark 3.1. For those not so familiar with the Hopf theorem we mention that these two theorems are essentially equivalent: for X a probability space, taking $\psi\equiv 1$ in (3.11) gives (3.10). Conversely, (3.11) can be proved for bounded functions from (3.10) by inducing on successively larger sets of finite measure; see [Fis92], and see [Zwe04] for the general L^1 case.

Let $D=D_{\mathbb{R}}$ denote two-sided Skorokhod path space, the càdlàg functions $f:\mathbb{R}\to\mathbb{R}$, let $D_0\subset D$ denote the elements f with $f(0)=0$, and let D_{0+} denote the subset of nondecreasing elements of D_0. Then there is a bijective correspondence from D_{0+} to the Borel measures on $\mathbb{R}$, defined by
$$\begin{aligned}\rho\big((0,t]\big)&=f(t)\quad\text{for } t\geq 0\\ \rho\big((t,0]\big)&=-f(t)\quad\text{for } t<0,\end{aligned}\ . \tag{3.12}$$
and extending additively from these intervals to the Borel sets; (3.12) defines, furthermore, a bijection between the elements of D_0 which are of local bounded variation and the charges (signed measures) on $\mathbb{R}$. We recall that a function of local bounded variation can be written as $f=g-h$ where g,h are nondecreasing functions (Theorem 4, §5.2 of [Roy68]); there is a way to make this decomposition canonical:

Definition 3.3. *Given a function $f \in D_0$ of local bounded variation, define a charge $\rho = \rho_f$ by* (3.12)*. Letting $\rho = \rho^+ - \rho^-$ be its Hahn decomposition (p. 236 of* [Roy68]*), so ρ^+, ρ^- are mutually singular measures, we define $f^+, f^- \in D_{0+}$ by reversing equation* (3.12) *for ρ^+, ρ^-. Then we call $f = f^+ - f^-$ the Hahn decomposition of f.*

Lemma 3.1. *Let Ψ be a real–valued cocycle over a measurable flow (X, μ, τ_t). Assume that for a.e. x, $\Psi(x,t)$ is a function of local bounded variation in t. Then the Hahn decomposition $\Psi(x,t) = \Psi^+(x,t) - \Psi^-(x,t)$ defines a pair of cocycles $\Psi^+(x,t), \Psi^-(x,t)$.*

Proof. The cocycle property implies $\rho_{\tau_s x}(A) = \rho_x(A + s)$; ρ_x^+, ρ_x^- satisfy this same equation, and reversing the logic, $\Psi^+(x,t)$ and $\Psi^-(x,t)$ defined from these measures are cocycles, with $\Psi = \Psi^+ - \Psi^-$. □

Here is our extension of the ergodic theorems to cocycles:

Theorem 3.2. *Let τ_t be a conservative, ergodic, measure–preserving flow on a σ–finite measure space (X, μ). Let Φ, Ψ be real–valued cocycles, measurable, of local bounded variation in t. Assume $\mathbb{I}(\Phi)$, $\mathbb{I}(\Psi) < \infty$ and $\mathbb{I}(\Phi) \neq 0$. Then, for $\mu-$a.e. $x \in X$,*

(i) (Cocycle Birkhoff Ergodic Theorem) if $\mu(X) = 1$:

$$\lim_{T\to\infty} \frac{1}{T}\Psi(x,T) = \mathbb{I}(\Psi). \tag{3.13}$$

(ii) (Cocycle Hopf Ergodic Theorem) For $0 < \mu(X) \leq +\infty$:

$$\lim_{T\to\infty} \frac{\Psi(x,T)}{\Phi(x,T)} = \frac{\mathbb{I}(\Psi)}{\mathbb{I}(\Phi)}. \tag{3.14}$$

Proof. We begin with (i). If Ψ were differentiable along orbits, we could prove this simply by applying the usual ergodic theorem to the derivative. However as in Example 1 above, that may not be the case. The idea of the proof will be to, instead, replace the derivative by a difference quotient, and apply the usual Birkhoff Ergodic Theorem for flows to that function.

Thus, for some fixed $a > 0$ we define

$$\Psi_a(x,t) \equiv \frac{1}{a}\big(\Psi(x,t+a) - \Psi(x,t)\big),$$

$$\Psi_{-a}(x,t) \equiv \frac{1}{a}\big(\Psi(x,t) - \Psi(x,t-a)\big).$$

Now $\Psi_a(x,0) = \frac{1}{a}\Psi(x,a)$ and $\Psi_{-a}(x,0) = \frac{1}{-a}\Psi(x,-a)$, so by (i) of Proposition 3.1,

$$\int_X \Psi_a(x,0)\mathrm{d}\mu(x) = \mathbb{I}(\Psi) = \int_X \Psi_{-a}(x,0)\mathrm{d}\mu(x), \tag{3.15}$$

$+\infty$ being allowed here. We have therefore, by the Birkhoff theorem for flows: for a.e. x,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \Psi_a(\tau_t x, 0)\ \mathrm{d}t = \int_X \Psi_a(x,0)\mathrm{d}\mu(x) = \mathbb{I}(\Psi) \tag{3.16}$$

and similarly for Ψ_{-a}.

Now we compare this to $\frac{1}{T}\Psi(x,T)$. For all $s\in\mathbb{R}$, since $\Psi_a(x,t)=\Psi_a(\tau_t(x),0)$, we have

$$\frac{1}{a}\int_s^{s+a}\Psi(x,t)\ \mathrm{d}t=\int_0^s\Psi_a(\tau_t(x),0)\ \mathrm{d}t+\frac{1}{a}\int_0^a\Psi(x,t)\ \mathrm{d}t\quad\text{and}\tag{3.17}$$

$$\frac{1}{a}\int_{s-a}^{s}\Psi(x,t)\ \mathrm{d}t=\int_0^s\Psi_{-a}(\tau_t(x),0)\ \mathrm{d}t+\frac{1}{a}\int_{-a}^0\Psi(x,t)\ \mathrm{d}t.\tag{3.18}$$

Since by Lemma 3.1 $\Psi(x,t)$ has a Hahn decomposition, we can assume without loss of generality that the cocycle Ψ is nondecreasing. We have, taking $s=T$ successively in (3.17) and (3.18):

$$\Psi(x,T)\le\frac{1}{a}\int_T^{T+a}\Psi(x,t)\ \mathrm{d}t=\int_0^T\Psi_a(\tau_t(x),0)\ \mathrm{d}t+\frac{1}{a}\int_0^a\Psi(x,t)\ \mathrm{d}t,$$

$$\int_0^T\Psi_{-a}(\tau_t(x),0)\ \mathrm{d}t+\frac{1}{a}\int_{-a}^0\Psi(x,t)\ \mathrm{d}t=\frac{1}{a}\int_{T-a}^{T}\Psi(x,t)\ \mathrm{d}t\le\Psi(x,T).$$

So

$$\Psi(x,-a)+\int_0^T\Psi_{-a}(\tau_t(x),0)\ \mathrm{d}t\le\Psi(x,T)\le\int_0^T\Psi_a(\tau_t(x),0)\ \mathrm{d}t+\Psi(x,a).\tag{3.19}$$

For x and a fixed, dividing by T, and using (3.16),

$$\mathbb{I}(\Psi)\le\lim_{T\to\infty}\frac{1}{T}\Psi(x,T)\le\mathbb{I}(\Psi).$$

Next we prove (ii). Given Hahn decompositions $\Psi=\Psi^+-\Psi^-$ and $\Phi=\Phi^+-\Phi^-$, we first prove the theorem under the assumption that all four cocycles have integral >0. First we consider the pair Ψ^+,Φ^+, assuming to simplify the notation that $\Psi=\Psi^+$ and $\Phi=\Phi^+$. From (3.19) we have

$$\frac{\Psi(x,-a)+\int_0^T\Psi_{-a}(\tau_t(x),0)\ \mathrm{d}t}{\Phi(x,a)+\int_0^T\Phi_a(\tau_t(x),0)\ \mathrm{d}t}\le\frac{\Psi(x,T)}{\Phi(x,T)}\le\frac{\int_0^T\Psi_a(\tau_t(x),0)\ \mathrm{d}t+\Psi(x,a)}{\int_0^T\Phi_{-a}(\tau_t(x),0)\ \mathrm{d}t+\Phi(x,-a)}.\tag{3.20}$$

From (3.11), and using (3.15), we have that

$$\lim_{T\to\infty}\frac{\int_0^T\Psi_{-a}(\tau_t(x),0)\ \mathrm{d}t}{\int_0^T\Phi_a(\tau_t(x),0)\ \mathrm{d}t}=\frac{\mathbb{I}(\Psi)}{\mathbb{I}(\Phi)},$$

and similarly for the upper bound; noting that $\int_0^T\Psi_a(\tau_t(x),0)\ \mathrm{d}t=\frac{1}{a}\int_0^T\Psi(x,a+t)\ \mathrm{d}t\to\infty$ as $T\to\infty$ for a.e. x, by (iv) of Proposition 3.1, and similarly for Ψ_{-a}, indeed (3.14) holds true.

The remaining pairs (Ψ^-,Φ^+), (Ψ^-,Φ^-), and (Ψ^+,Φ^-) are handled in the same way. Next we put this information together. Now, since all four functions are eventually nonzero, we have

$$\lim_{T\to\infty}\frac{\Psi(x,T)}{\Phi^+(x,T)}=\lim_{T\to\infty}\frac{\Psi^+(x,T)}{\Phi^+(x,T)}-\lim_{T\to\infty}\frac{\Psi^-(x,T)}{\Phi^+(x,T)}=\frac{\mathbb{I}(\Psi^+)}{\mathbb{I}(\Phi^+)}-\frac{\mathbb{I}(\Psi^-)}{\mathbb{I}(\Phi^+)}=\frac{\mathbb{I}(\Psi)}{\mathbb{I}(\Phi^+)},$$

which implies (3.14) for (Ψ,Φ^+) hence also for the reciprocal pair (Φ^+,Ψ); similarly, we have this for the pair (Φ^-,Ψ) and thus (3.14) holds for (Φ,Ψ) and hence for (Ψ,Φ). Finally if some of the integrals are zero (but with $\mathbb{I}(\Phi)$ eventually nonzero), the argument is yet simpler, so we are done. □

We illustrate the cocycle Birkhoff theorem with a case where the usual theorem does not directly apply:

Proposition 3.2. *Let (X, μ, τ_t) be an ergodic flow on a probability space and $B \subseteq X$ a measurable, finite measure cross–section with measure μ_B. For the cocycle $N_B(x,t)$ of Example 1, then for μ_B–a.e. $x \in B$, and also for μ–a.e. $x \in X$, $\lim_{T\to\infty} \frac{1}{T} N_B(x,T) = 1/\mathbb{E}(r)$.*

Proof. From Theorem 3.2 we know that for $\Psi(x,t) = N_B(x,t)$, then $\lim_{T\to\infty} \frac{1}{T}\Psi(x,T) = \mathbb{I}(\Psi)$ for μ–a.e. x. Now using (iii) of Proposition 3.1, $\mathbb{I}(\Psi) = \int_B \Psi(x, r(x)) \mathrm{d}\mu_B(x) = \int_B 1 \mathrm{d}\mu_B(x) = \mu_B(B)$. On the other hand, $\mathbb{E}(r) = \int_B r(x) \mathrm{d}\frac{\mu_B}{\mu_B(B)}(x) = 1/\mu_B(B)$, finishing the proof. (The statement for a.e. x in the cross–section then follows since ergodic averages are constant along flow orbits). □

4. The scaling and increment flows and their duals

In this section we consider certain topological and Borel measurable flows on two–sided Skorokhod space $D = D_{\mathbb{R}}$ endowed with the J_1–topology; see [FT12] for background and references.

We write $D_{\geq} \subset D$ for the nondecreasing functions which go to $+\infty$ at $+\infty$ and $-\infty$ at $-\infty$, and $D_{>}$ for the subset of $D_{\geq}$ of increasing functions. Recall that the *generalized inverse* of $f \in D$ is $\widehat{f}(t) = \inf\{s : f(s) > t\}$. We write $\mathcal{I}(f) = \widehat{f}$, and speak of the path $\widehat{f}$ as being *dual* to f.

Note that $\mathcal{I} : D_{\geq} \to D_{\geq}$. Let $D_{0\geq} = D_0 \cap D_{\geq}$ and $D_{0>} = D_0 \cap D_{>}$.

We note that:

Lemma 4.1. *The map $\mathcal{I} : D \to D$ is a Borel measurable involution. It maps $D_{\geq}$ to $D_{\geq}$. The image $\widehat{D}_{0>} \equiv \mathcal{I}(D_{0>})$ is the collection of continuous elements of $D_{0\geq}$ such that 0 is a point of right increase; jump points of f become flat spots for $\widehat{f}$.* □

4.1. Two pairs of flows.

Choosing $\beta > 0$, the *scaling flow* τ_t of index β on D is defined by

$$(\tau_t f)(x) = \frac{f(e^t x)}{e^{\beta t}}.$$

The *increment flow* η_t is defined by

$$(\eta_t f)(x) = f(x+t) - f(t).$$

Proposition 4.1. *The scaling flow is J_1–continuous, i.e. it is a jointly continuous function from $D \times \mathbb{R}$ to D; the increment flow is (jointly, Borel) measurable. The pair of flows satisfies the following commutation relation: for all $s,t \in \mathbb{R}$,*

$$\tau_t \circ \eta_s = \eta_{e^{-t}s} \circ \tau_t. \tag{4.1}$$

Proof. It is easy to check (4.1); for the continuity and measurability of τ_t see [FT12] and [FT11]. The σ–algebras generated by the J_1– and uniform–on–compact topologies on D are the same. Moreover this agrees with the restriction to D of the Kolmogorov σ–algebra, that generated by the product topology on $\mathbb{R}^{\mathbb{R}}$, i.e. by the coordinate projections; the reason is that since a path in D has left and right limits, the uniform neighborhood of a path is determined by a coordinate neighborhood at a countable dense set of points. One checks that the increment flow η_t on $\mathbb{R}^{\mathbb{R}}$ is Borel measurable for the Kolmogorov σ–algebra, a fortiori for its restriction to D. □

(Regarding equality of the σ–algebras and further references see also Theorem 11.5.2 of [Whi02].)

Fixing now $\alpha > 0$, we let τ_t denote the scaling flow of index $1/\alpha$ acting on $D_{0>}$ and $\widehat{\tau}_t$ the scaling flow of index α on $\widehat{D}_{0>}$; η_t is the increment flow on $D_{0>}$, while $\widehat{\eta}_t$ on $\widehat{D}_{0>}$ is defined by duality:

$$\widehat{\eta}_t \widehat{f} = \widehat{(\eta_t f)}.$$

The increment flow on the dual space $\widehat{D}_{0>}$ is denoted $\overline{\eta}_t$. We call $\widehat{\eta}_t$ the *increment subflow*, as it flows along a singular subset of times of $\overline{\eta}_t$, skipping over flat intervals of the path $\widehat{f}$.

Proposition 4.2. (i) *We have these topological isomorphisms of flows:*

$$\widehat{\tau}_{t/\alpha} \circ \mathcal{I} = \mathcal{I} \circ \tau_t \text{ and } \widehat{\eta}_t \circ \mathcal{I} = \mathcal{I} \circ \eta_t.$$

(ii) $\widehat{\tau}_t$ *is continuous,* $\widehat{\eta}_t$ *is Borel measurable, and they satisfy the commutation relation*

$$\widehat{\tau}_t \circ \widehat{\eta}_s = \widehat{\eta}_{e^{-\alpha t}s} \circ \widehat{\tau}_t \qquad \forall s, t.$$

Proof. To prove (i), we write

$$\widehat{\tau_t f}(x) = \inf\{u : \tau_t f(u) > x\} = \inf\{u : f(e^t u) > x\, e^{t/\alpha}\} = \frac{1}{e^t}\widehat{f}(x\, e^{t/\alpha}) = \widehat{\tau}_{t/\alpha}\widehat{f}(x).$$

By Lemma 4.1, $\widehat{\eta}_s$ is a Borel measurable flow. For the commutation relation of $\widehat{\tau}_t, \widehat{\eta}_s$ we use (i) together with (4.1), as for all $f \in \widehat{D}_{0>}$,

$$\widehat{\tau}_t(\widehat{\eta}_s \widehat{f}) = \widehat{\tau}_t(\widehat{\eta_s f}) = (\widehat{\tau_{\alpha t}\eta_s f}) = (\widehat{\eta_{e^{-\alpha t}s}\, \tau_{\alpha t} f}) = \widehat{\eta}_{e^{-\alpha t}s}(\widehat{\tau_{\alpha t} f}) = \widehat{\eta}_{e^{-\alpha t}s}(\widehat{\tau}_t \widehat{f}).$$

□

The relationship between η_t and $\overline{\eta}_t$ is best expressed in terms of the *completed graph* Γ_f of f, see also [FT11]:

$$\Gamma_f = \{(x, y) \in \mathbb{R} \times \mathbb{R} : \ f(x^-) \le y \le f(x) \equiv f(x^+)\},$$

where $f(x^-)$ (resp. $f(x^+)$) stands for the limit from the left (resp. right) at x. Defining the *dual graph* $\widehat{\Gamma}_f = \{(y, x) : \ (x, y) \in \Gamma_f\}$, the completed graph of the generalized inverse is $\Gamma_{\widehat{f}} = \widehat{\Gamma}_f$.

Lemma 4.2. *The increment flow* $(D_{0>}, \eta_t)$ *and the dual increment flow* $(\widehat{D}_{0>}, \overline{\eta}_t)$ *are related as follows. For* $f \in D_{0>}$ *we have:*

$$\widehat{\Gamma}_{\eta_t f} = \Gamma_{\overline{\eta}_{f(t)}\widehat{f}}$$

Proof. Since $f \in D_{0>}$, every t is a point of right increase of f, so $\widehat{f}(f(t)) = t$. Now, η_t acts on the completed graph via:

$$\Gamma_{\eta_t f} = \Gamma_f - \big(t, f(t)\big), \quad \text{so}$$

$$\widehat{\Gamma}_{\eta_t f} = \big(\widehat{\Gamma_f - (t, f(t)}\big) = \widehat{\Gamma}_f - (f(t), t) = \Gamma_{\widehat{f}} - \big(f(t), \widehat{f}(f(t))\big) = \Gamma_{\overline{\eta}_{f(t)}\widehat{f}}.$$

□

Corollary 4.1. *Let* $B \subseteq D_{0>}$ *satisfies that the return–time function* $r(f) = \min\{t > 0 : \ \eta_t(f) \in B\}$ *exists for all* $f \in B$ *(and so is everywhere finite and strictly positive). Then* $\widehat{B}$ *satisfies this for the increment subflow* $\widehat{\eta}_t$ *with return–time function* $\widehat{r}(\widehat{f}) = r(f)$*, and also for the dual increment flow* $\overline{\eta}_t$*, with return–time function* $\overline{r}(\widehat{f}) = f(r(f)) = f(\widehat{r}(\widehat{f}))$.

Proof. For $\widehat{\eta}_t$ this follows from the isomorphism with η_t. Indeed,

$$\widehat{r}(\widehat{f}) = \min\{t > 0 : \widehat{\eta}_t \widehat{f} \in \widehat{B}\} = \min\{t > 0 : \widehat{\eta_t f} \in \widehat{B}\} = r(f).$$

For $\overline{\eta}_t$, say for some $f \in B$, $a = r(f)$. Then $\eta_a f = g \in B$. From Lemma 4.2,

$$\widehat{\Gamma}_{\eta_a f} = \widehat{\Gamma}_g = \Gamma_{\overline{\eta}_b \widehat{f}}$$

for $b = f(a) = f(r(f))$. This is the least such time and so is the return time for the flow $\overline{\eta}_t$. □

5. Measures for the increment flow

We now study invariant measures for the increment flow, beginning with the dual flow of §4. The starting point will be the general framework of ergodic self–similar processes with stationary increments. Then in §5.1 we specialize to completely asymmetric stable processes of index $\alpha \in (0,1)$ and their duals, the Mittag–Leffler processes. In §5.2 we treat the renewal flow, and in §5.3 we examine the special case of integer gaps.

Definition 5.1. *By an* ergodic self–similar process $X(t)$ of index $\beta > 0$ *and with paths in D we mean that we have a probability measure ν on $D = D_{\mathbb{R}}$ (resp. $D_{\mathbb{R}^+}$) a* two–sided *(resp.* one–sided*) process which is invariant and ergodic for the scaling flow of index β. We say the process has* stationary increments *iff this same measure is invariant for the increment flow $\{\eta_t\}_{t\in\mathbb{R}}$ (resp.* semi*flow $\{\eta_t\}_{t\geq 0}$) in the case of $D_{\mathbb{R}}$ (resp. $D_{\mathbb{R}^+}$).*

Remark 5.1. The definitions of self–similar process, or a process with stationary increments, can equivalently be formulated for finite cylinder sets since these uniquely determine the measure on D. A one–sided process which is $\{\eta_t\}_{t\geq 0}$–invariant has a unique extension to a two–sided $\{\eta_t\}_{t\in\mathbb{R}}$–invariant process: one first extends via η_T–invariance for fixed $T < 0$ to the Borel σ–algebra on $D_{\mathbb{R}}$ over times in the interval $[T, \infty)$, then notes that the nested union as $T \to -\infty$ of these generates the full σ–algebra of $D_{\mathbb{R}}$. If the one–sided process is self–similar, then so is the two–sided process. Note that self–similarity forces $X(0) = 0$ a.s.

Now assume we are given a two–sided process which is self–similar with stationary increments, with paths in $D_{0>}$; we denote by ν the probability measure on path space $D_{0>}$, which is invariant and ergodic for the flows τ_t and η_t. Via Proposition 4.2 the pushed–forward measure $\widehat{\nu} = \mathcal{I}^*(\nu)$ on the dual space $\widehat{D}_{0>}$ is invariant for $\widehat{\tau}_t$ and for the increment subflow $\widehat{\eta}_t$.

We next define from this a measure $\overline{\nu}$ which will be invariant for the dual increment flow $\overline{\eta}_t$ itself, and which may be infinite.

To carry this out, we first define functions $\Psi : D_{0>} \times \mathbb{R} \to \mathbb{R}$ and $\Phi, \widetilde{\Phi}, \widehat{\Psi} : \widehat{D}_{0>} \times \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} \Psi(f,t) &= f(t), & \Phi(\widehat{f},t) &= \widehat{f}(t), \\ \widetilde{\Phi}(\widehat{f},t) &= \widehat{f}(f(t)), & \widehat{\Psi}(\widehat{f},t) &= f(t). \end{aligned}$$

Proposition 5.1. *The functions Ψ and Φ are cocycles over the flows $\eta_t, \overline{\eta}_t$ respectively, and $\widetilde{\Phi}$ and $\widehat{\Psi}$ are cocycles over $\widehat{\eta}_t$.*

Proof. That Φ, Ψ and $\widehat{\Psi}$ are cocycles follows from the definitions. We only say a word about $\widetilde{\Phi}$: since for $f \in D_{0>}$ every t is a point of increase, $\widetilde{\Phi}(\widehat{f},t) = t$. And so

$$\widetilde{\Phi}(\widehat{f}, t+s) = t+s = \widetilde{\Phi}(\widehat{f},t) + \widetilde{\Phi}(\widehat{\eta_t f}, s) = \widetilde{\Phi}(\widehat{f},t) + \widetilde{\Phi}(\widehat{\eta}_t\widehat{f}, s)$$

so this is a cocycle over $\widehat{\eta}_t$. $\square$

Next we define the promised invariant measure for $\overline{\eta}_t$.

First, since the flow η_t preserves the probability measure ν, it is conservative (by the Poincaré recurrence theorem); it is futhermore assumed ergodic, and so we can apply the Ambrose–Kakutani theorem, choosing a finite measure cross–section (B, ν_B, T), with first return map T and return–time function $r : B \to (0, +\infty)$. Setting $\widehat{T}(\widehat{f}) \equiv \widehat{(Tf)}$ and $\widehat{\nu}_{\widehat{B}} = \mathcal{I}^*(\nu_B)$, by the isomorphism of η_t with

$\widehat{\eta}_t$ we have that $(\widehat{B}, \widehat{\nu}_B, \widehat{T})$ is a cross–section for the flow $(\widehat{D}_{0>}, \widehat{\eta}_t, \widehat{\nu})$, with return time $\widehat{r}(\widehat{f}) = r(f)$. We then form a second special flow over $(\widehat{B}, \widehat{\nu}_B, \widehat{T})$, choosing a different return–time function.

Proposition 5.2. *The special flow over $(\widehat{B}, \widehat{\nu}_B, \widehat{T})$ with return time $\overline{r}(\widehat{f}) = \widehat{\Psi}(\widehat{f}, \widehat{r}(\widehat{f}))$ is a special flow representation for $\overline{\eta}_t$; indeed, for $\widehat{\nu}_B$–a.e. $\widehat{f} \in \widehat{B}$, the return time for $\overline{\eta}_t$ to $\widehat{B}$ is $\overline{r}(\widehat{f})$.*

Proof. But this is now immediate from Corollary 4.1. □

We transport the special flow measure to a measure written $\overline{\nu}$ on the dual increment flow $(\widehat{D}_{0>}, \overline{\eta}_t)$. The next step will be to calculate its total mass; for that we first show:

Proposition 5.3. *The integrals of the cocycles are $\mathbb{I}(\Psi) = \mathbb{I}(\widehat{\Psi}) = \int_{D_{0>}} f(1)\,\mathrm{d}\nu(f)$ and $\mathbb{I}(\Phi) = \mathbb{I}(\widetilde{\Phi}) = 1$.*

Proof. The integral of the cocycle Ψ is, by (i) of Proposition 3.1,

$$\mathbb{I}(\Psi) = \int_{D_{0>}} \Psi(f,1)\,\mathrm{d}\nu(f) = \int_{D_{0>}} f(1)\,\mathrm{d}\nu(f).$$

Similarly,

$$\mathbb{I}(\widehat{\Psi}) = \int_{\widehat{D}_{0>}} \widehat{\Psi}(\widehat{f},1)\,\mathrm{d}\widehat{\nu}(\widehat{f}) = \int_{\widehat{D}_{0>}} f(1)\,\mathrm{d}\widehat{\nu}(\widehat{f}) = \int_{D_{0>}} f(1)\,\mathrm{d}\nu(f).$$

To evaluate the integral of Φ, we choose, as in Proposition 5.2, a finite measure cross–section $\widehat{B}$ for $\widehat{\eta}_t$ with return–time function $\widehat{r}$. This is also a cross–section for $\overline{\eta}_t$, with return time $\overline{r}(\widehat{f})$ and with equal base measure $\overline{\nu}_{\widehat{B}} = \widehat{\nu}_{\widehat{B}}$. We have, using Proposition 3.1 (iii),

$$\mathbb{I}(\Phi) = \int_{\widehat{D}_{0>}} \Phi(\widehat{f},1)\,\mathrm{d}\overline{\nu}(\widehat{f}) = \int_{\widehat{B}} \Phi(\widehat{f},\overline{r}(\widehat{f}))\,\mathrm{d}\overline{\nu}_{\widehat{B}}(\widehat{f}) = \int_{\widehat{B}} \widehat{f}(\overline{r}(\widehat{f}))\,\mathrm{d}\overline{\nu}_{\widehat{B}}(\widehat{f}) = \int_{\widehat{B}} \widehat{f}(\overline{r}(\widehat{f}))\,\mathrm{d}\widehat{\nu}_{\widehat{B}}(\widehat{f}).$$

Now by Corollary 4.1, $\overline{r}(\widehat{f}) = f(\widehat{r}(\widehat{f}))$, so $\widehat{f}(\overline{r}(\widehat{f})) = \widehat{f}(f(\widehat{r}(\widehat{f}))) = \widehat{r}(\widehat{f})$ and this is

$$\int_{\widehat{B}} \widehat{r}(\widehat{f})\,\mathrm{d}\widehat{\nu}_{\widehat{B}}(\widehat{f}) = \widehat{\nu}(\widehat{D}_{0>}) = 1.$$

And lastly,

$$\mathbb{I}(\widetilde{\Phi}) = \int_{\widehat{B}} \widetilde{\Phi}(\widehat{f},\widehat{r}(\widehat{f}))\,\mathrm{d}\widehat{\nu}_{\widehat{B}}(\widehat{f}) = \int_{\widehat{B}} \widehat{r}(\widehat{f})\,\mathrm{d}\widehat{\nu}_{\widehat{B}}(\widehat{f}) = 1.$$

□

Corollary 5.1. *The dual increment flow $(\widehat{D}_{0>}, \overline{\nu}, \overline{\eta}_t)$ is conservative, and it is ergodic iff the increment flow $(D_{0>}, \nu, \eta_t)$ is; the total mass for the invariant measure $\overline{\nu}$ is $\overline{\nu}(\widehat{D}_{0>}) = \int_{D_{0>}} f(1)\,\mathrm{d}\nu(f)$.*

Proof. Again, being conservative is automatic when the space has finite measure; hence the cross–section map is recurrent. Ergodicity and recurrence then pass from the cross–section map to the flow.

The total mass of the special flow is

$$\int_{\widehat{B}} \overline{r}(\widehat{f})\,\mathrm{d}\widehat{\nu}_{\widehat{B}}(\widehat{f}) = \int_{\widehat{B}} \widehat{\Phi}(\widehat{f},\widehat{r}(\widehat{f}))\,\mathrm{d}\widehat{\nu}_{\widehat{B}}(\widehat{f}) = \mathbb{I}(\widehat{\Phi}) = \mathbb{I}(\Psi) = \int_{D_{0>}} f(1)\,\mathrm{d}\nu(f)$$

by (iii) of Proposition 3.1 together with Proposition 5.3. □

5.1. **A measure for the increment flow of the Mittag–Leffler process.** Here Z denotes a positive and completely asymmetric two–sided α–stable process ($\alpha \in (0,1)$); its law ν lives on $D_{0>}$ and is invariant for τ_t the scaling flow of index $1/\alpha$. Its generalized inverse $\widehat{Z}$, with law $\widehat{\nu}$ invariant for the scaling flow $\widehat{\tau}_t$ of index α, lives on $\widehat{D}_{0>}$, while the measure $\overline{\nu}$ lives on $D_{0\geq}$. We write $\overline{\eta}_t$, $\widehat{\eta}_t$ for the increment flow and subflow respectively.

Proposition 5.4. *The pairs of flows τ_t, η_t on $(D_{0>}, \nu)$, and $\widehat{\tau}_t, \widehat{\eta}_t$ on $(\widehat{D}_{0>}, \widehat{\nu})$ are Bernoulli flows of infinite entropy. They satisfy the commutation relations and properties in Propositions 4.1 and 4.2. The pair $\widehat{\tau}_t, \overline{\eta}_t$ satisfies the commutation relation of Proposition 4.1; $(D_{0\geq}, \overline{\nu}, \overline{\eta}_t)$ is a conservative ergodic flow, with $\overline{\nu}$ a σ–finite, infinite measure.*

Proof. That τ_t is a Bernoulli flow of infinite entropy on the Lebesgue space (D, ν) is proved in Lemma 3.3 of [FT12]. By Proposition 4.2, the flows $(D, \widehat{\nu}, \widehat{\tau}_{t/\alpha})$ and (D, ν, τ_t) are isomorphic, so the same holds for $\widehat{\tau}_t$.

Next we consider the increment flows. Since the stable process Z has stationary increments, η_t preserves ν. Since these increments are, moreover, independent, $(X_n)_{n\in\mathbb{Z}}$ with $X_n \equiv Z(n+1) - Z(n)$ is an i.i.d. sequence, with law $G_{\alpha,1}$. This is an infinite entropy Bernoulli shift; its path space is $\Pi = \Pi_{-\infty}^{+\infty}\mathbb{R}$ with infinite product measure μ_α of distribution $G_{\alpha,1}$ on each coordinate and left shift map σ. Now $X_n = (\eta_n(Z))(1)$, so $Z \mapsto (X_n)_{n\in\mathbb{Z}}$ defines a measure–preserving map from (D, ν, η_1) to $(\Pi, \mu_\alpha, \sigma)$. That is, the time–one map η_1 has a homomorphic image which is a Bernoulli shift of infinite entropy. The same is true for each time $1/k$ map for $n = 1, 2, \ldots$. These increasing σ–algebras of Borel sets separate points of D, hence generate the Borel σ–algebra, so η_1 is itself indeed a Bernoulli shift and (D, ν, η_t) is a Bernoulli flow.

Now for $\alpha \in (0,1)$ and Z positive and completely asymmetric by Propositions 4.2 and 5.4 the flow (D, ν, η_t) is isomorphic to $(D, \widehat{\nu}, \widehat{\eta}_t)$, hence this second flow is also infinite entropy Bernoulli.

As we saw above, the fact that the flows $(D, \widehat{\nu}, \widehat{\eta}_t)$ and $(D, \overline{\nu}, \overline{\eta}_t)$ share a common cross–section allows the ergodicity of $\widehat{\eta}_t$ to pass over to recurrence and ergodicity for the infinite measure flow $\overline{\eta}_t$. Lastly, from Corollary 5.1, the total mass is $\overline{\nu}(\widehat{D}_{0>}) = \mathbb{E}(Z(1)) = +\infty$, as $0 < \alpha < 1$. $\square$

5.2. **A measure for the renewal flow.** We fix a probability measure μ_F on $\mathbb{R}^+$ with distribution function F and place the infinite product measure $\mu = \otimes_{-\infty}^{\infty}\mu_F$ on $B \equiv \Pi_{-\infty}^{\infty}\mathbb{R}^+$. The left shift map σ acts on the Lebesgue space (B, μ), with points $x = (\ldots x_{-1}x_0x_1\ldots)$. Then (B, μ, σ) is a Bernoulli shift (in the generalized sense: the "state space" is the support of μ_F and may be uncountable). We define the *renewal flow* (X, ρ, h_t) to be the special flow over (B, μ, σ) with return–time function $r(x) = x_0$, see §3.1.

Next we represent the renewal flow as an increment flow on the nondecreasing paths $D_{0\geq}$; this will make precise the relationship between the renewal flow and renewal process. For this we define a map $\zeta : X \to D_{0\geq}$ by $\zeta(x) \equiv N_B(x, \cdot)$, where N_B counts the number of returns of the flow h_s to the cross–section as in Example 1, equation (3.6). We write $\overline{\mu}$ for the pushed–forward measure $\zeta^*(\rho)$, and $\widehat{\mu}$ for the measure pushed forward from μ on the cross–section B. We denote by $\overline{\eta}_t$ the increment flow on $D_{0\geq}$ (extending the previous definition from $\widehat{D}_{0>} \subseteq D_{0\geq}$).

We have:

Proposition 5.5. *The renewal flow (X, ρ, h_t) is conservative ergodic. The map ζ gives a flow isomorphism from the renewal flow (X, ρ, h_t) to the increment flow $(D_{0\geq}, \overline{\mu}, \overline{\eta}_t)$. The total mass of the flow is $\rho(X) = \int_{\mathbb{R}^+} s\,\mathrm{d}F(s)$.*

Proof. Since the base map is conservative ergodic (it is a Bernoulli shift), so is the flow. The map ζ is a bijection; that it is a conjugacy, i.e. $\zeta \circ h_t = \overline{\eta}_t \circ \zeta$, follows from the definition of the increment flow. From (3.2), the total mass for the flow (X, ρ, h_t) is $\int_B x_0 \,\mathrm{d}\mu(x) = \int_{\mathbb{R}^+} s \,\mathrm{d}F(s)$. $\square$

Note that we have defined two different invariant measures for the same flow, the increment flow $\overline{\eta}_t$ on $D_{0\geq}$: $\overline{\nu}$ for the two–sided Mittag–Leffler process and $\overline{\mu}$ for the renewal process.

5.3. **The case of integer gaps.** For the special case of a renewal process with integer gaps of distribution function F, we describe in this section an alternate construction of the renewal flow as the *suspension flow* over a map called the *renewal transformation* defined by F, that is, the special flow over that transformation with constant return time one. We begin by presenting several equivalent models for this map.

The gap distribution μ_F lives on $\mathbb{N}^* \subseteq \mathbb{R}^+$. For a first model of our map, we define $\mu = \otimes_{-\infty}^{\infty} \mu_F$ on $\Sigma = \Pi_{-\infty}^{+\infty} \mathbb{N}^*$, so (Σ, μ, σ) is a Bernoulli shift with this countable alphabet. With $x = (\dots x_{-1} x_0 x_1, \dots) \in \Sigma$, and $B_k = \{x : x_0 = k\}$, we define the tower transformation $(\widehat{\Sigma}, \widehat{\mu}, T)$ with base (Σ, σ, μ) and return time k over B_k. Thus, $\widehat{\Sigma} = \{(x, j) : 0 \leq j \leq k-1\} \subseteq \Sigma \times \mathbb{N}$ with map $T((x,j)) = (x, j+1)$ if $j < k-1$ and $T((x,j)) = (\sigma(x), 0)$ if $j = k-1$. The tower measure $\widehat{\mu}$ is the restriction of the product of μ with counting measure on $\mathbb{N}$ to $\widehat{\Sigma}$; note that the $\widehat{\mu}$–measure of the tower base is 1. We call $(\widehat{\Sigma}, \widehat{\mu}, T)$ the *tower model* of the renewal transformation.

For a second model which we term the *event process*, let $\psi = \chi_B$ be the indicator function of the tower base $B = \Sigma$, and given a point (x, k) in $\widehat{\Sigma}$, let $(Y_i)_{i \in \mathbb{Z}} \in \Pi_Y \equiv \Pi_{-\infty}^{+\infty}\{0, 1\}$ be the sequence $Y_i = Y_i(x, k) = \chi_B \circ T^i((x, k))$. This defines a function from $\widehat{\Sigma}$ to Π_Y which conjugates T to the shift map σ. We write μ_Y for the push-forward of $\widehat{\mu}$, and note that the μ_Y–measure of $[Y_0 = 1]$ is 1.

For a third model, we define $\Psi((x, k), n)$ to be the cocycle over the tower model generated by $\psi = \chi_B$ (see (3.4)). Writing $\widetilde{N}_n(x, k) \equiv \Psi((x, k), n)$, let $\widehat{\mu}_{\widetilde{N}}$ denote the pushed–forward measure on $\Pi_{\widetilde{N}} \equiv \Pi_{-\infty}^{+\infty} \mathbb{Z}$, via the map which sends (x, k) to $\widetilde{N}_n = \widetilde{N}_n(x, k)$. Then the *increment shift* $(\Theta(\widetilde{N}))_n = \widetilde{N}_{n+1} - \widetilde{N}_1$ preserves $\widehat{\mu}_{\widetilde{N}}$.

The fourth model is a countable state Markov shift, the *renewal shift.* The state space is $\mathbb{N}^*$; for the probability vector $p = (p_k)_{k \geq 1}$ on $\mathbb{N}^*$ with $p_k = \mu_F(k)$, we define the transition matrix

$$P = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \dots \\ 1 & 0 & 0 & 0 \dots \\ 0 & 1 & 0 & 0 \dots \\ 0 & 0 & 1 & 0 \dots \\ \vdots & & & \end{bmatrix}$$

The shift–invariant Markov measure $\widehat{\mu}_P$ on $\Pi = \Pi_{-\infty}^{+\infty} \mathbb{N}^*$ is then defined in the usual way from the transition matrix plus an invariant row vector $\pi \equiv (\pi_1, \pi_2, \dots)$. As $\pi P = \pi$, choosing $\pi_1 = 1$ determines this vector uniquely: $\pi_k = \sum_{i \geq k}^{\infty} p_i$.

We have:

Proposition 5.6. *These four models of renewal transformation with integer gap distribution F are measure theoretically and topologically isomorphic:*
(i) the tower $(\widehat{\Sigma}, \widehat{\mu}, T)$ over the Bernoulli shift (Σ, μ, σ), with heights equal to the gaps;
(ii) the left shift map on the event process, (Π_Y, μ_Y, σ);
(iii) the increment shift Θ on $(\Pi_{\widetilde{N}}, \widehat{\mu}_{\widetilde{N}})$;

(iv) the renewal shift $(\Pi, \widehat{\mu}_P, \sigma)$ *with transition matrix* P *where* $p_k = \mu_F(k)$*, and with initial vector* $\pi \equiv (\pi_1, \pi_2, \dots)$ *given above. The total mass of the renewal transformation equals*

$$\sum_{k=1}^{\infty} \pi_k = \sum_{k=1}^{\infty} \sum_{i=k}^{\infty} p_i = \sum_{k=1}^{\infty} k p_k = \int_0^{\infty} s \, \mathrm{d}F(s).$$

The renewal flow is measure theoretically and topologically isomorphic to the suspension flow over the renewal transformation, and its total mass is that of the transformation.

Proof. The renewal flow was constructed above as a special flow over the Bernoulli shift on $\Pi_{-\infty}^{+\infty}\mathbb{R}^+$ with independent product measure of the distribution F and with return–time function $r(x) = x_0 = k$. Since F is supported on the integers, we can replace $\Pi_{-\infty}^{+\infty}\mathbb{R}^+$ by $\Pi_{-\infty}^{+\infty}\mathbb{N}^*$, which is the Bernoulli shift space for the base of the tower model. Since the return time for the tower is the same as for the flow, the flow is the supension flow of this map. Now the mass of the suspension flow is on the one hand given in Proposition 5.5, as $\int_0^{\infty} s \, \mathrm{d}F(s)$; on the other hand it equals that of its base map (the tower model).

The renewal transformation $(\widehat{\Sigma}, \widehat{\mu}, T)$ is naturally isomorphic to the event process shift (Π_Y, μ_Y, σ), since the correspondence $(x, k) \mapsto Y_i(x, k)$ is one–to-one, as cylinder sets of the event process determine the return–time partition of the base which in turn defines the Bernoulli shift.

For each (x, k), $\widetilde{N}_0 = 0$ and $\widetilde{N}_n = \Sigma_{i=0}^{n-1} Y_i$ for $n > 0$, $\widetilde{N}_n = \Sigma_{i=n}^{-1} Y_i$ for $n < 0$. This defines a map from Π_Y to $\Pi_{\widetilde{N}}$, which conjugates the shift to the increment shift and which is a bijection as $Y_n = \widetilde{N}_{n+1} - \widetilde{N}_n$ gives the inverse.

We define a map from the renewal shift to the tower map, sending first the set $\widetilde{B}_k \equiv \{x \in \Pi : x_0 = 1 \text{ and } x_1 = k+1\}$ to $B_k \times \{0\}$; these have the same measure, p_k. Then $\{x : x_0 = 1\} = \cup_{k \geq 1} \widetilde{B}_k$ maps to the base $\cup_{k \geq 1} B_k$, and both $\widetilde{B}_k$ and B_k have the same return time k to $\widetilde{B}$, B, hence the dynamics is conjugated. The independence of the gaps (the Markov property) is registered in the Bernoulli property of the base map, so the measures correspond. Choice of $\pi_1 = 1$ corresponds to the tower base having measure 1.

We mention that the correspondence from the renewal shift to the event process sends, e.g., the sequence $(\dots 154321321 \dots)$ to $(\dots 100001001 \dots)$, also clearly a bijection.

Now we have seen in the first paragraph that the mass of the tower model equals $\int_0^{\infty} s \, \mathrm{d}F(s)$. On the other hand, it can be calculated by adding up the measure of the column over each B_k, giving $\widehat{\mu}(\widehat{\Sigma}) = \sum_{k=1}^{\infty} k \mu(B_k) = \sum_{k=1}^{\infty} k p_k$. Calculated for the renewal shift, the total mass is that of its invariant vector: $\sum_{k=1}^{\infty} \pi_k = \sum_{k=1}^{\infty} \sum_{i=k}^{\infty} p_i = \sum_{k=1}^{\infty} k p_k$. □

6. Order–two ergodic theorems

6.1. **Order–two ergodic theorems for self–similar processes.** We begin in the context of a self–similar process, dual to a self–similar process with stationary increments.

Theorem 6.1. *For* $\alpha > 0$*, let* ν *be a probability measure on* $D_{0>}$*, invariant and ergodic for the scaling flow* τ_t *of index* $1/\alpha$ *and also for the increment flow* η_t*. Let* $\widehat{\tau}_t$ *denote the dual scaling flow of index* α *on* $(\widehat{D}_{0>}, \widehat{\nu})$*, and* $(\widehat{D}_{0>}, \widehat{\nu}, \widehat{\eta}_t)$ *the increment subflow. We write* $\overline{\nu}$ *for the* $\overline{\eta}_t$*- invariant measure defined in the first part of* §5*. Let* $\Phi(\widehat{f}, t)$ *be a cocycle for the increment flow* $\overline{\eta}_t$*which is*

jointly measurable, of local bounded variation in t, with $\mathbb{I}(\Phi)$ finite. Then for $\overline{\nu}-$a.e. $\widehat{f}$ we have:

$$\lim_{T\to\infty}\frac{1}{\log T}\int_1^T \frac{\Phi(\widehat{f},t)}{t^\alpha}\frac{\mathrm{d}t}{t} = c_0\,\mathbb{I}(\Phi), \tag{6.1}$$

with $c_0 \equiv \int_{\widehat{D}_{0>}} \widehat{f}(1)\mathrm{d}\widehat{\nu}(\widehat{f})$.

Proof. By Corollary 5.1, since $\widehat{\eta}_t$ is ergodic, $(\widehat{D}_{0>}, \overline{\nu}, \overline{\eta}_t)$ is conservative ergodic, with total mass $\overline{\nu}(\widehat{D}_{0>}) = \int_{D_{0>}} f(1)\mathrm{d}\nu(f)$. Hence from the Hopf Theorem (3.14), it is enough to prove (6.1) for some specific cocycle Φ over $\overline{\eta}_t$ with $\mathbb{I}(\Phi) \neq 0$. We choose

$$\Phi(\widehat{f},t) \equiv \widehat{f}(t).$$

Using Propositions 5.1 and 5.3, Φ is a cocycle over $\overline{\eta}_t$, with integral $\mathbb{I}(\Phi) = 1$.

First we prove (6.1) for $\widehat{\nu}$–a.e. $\widehat{f}$, i.e. with respect to this finite measure. For our choice of Φ, after a change of variables, we have:

$$\frac{1}{R}\int_0^R \frac{\widehat{f}(e^u)}{e^{\alpha u}}\,\mathrm{d}u = \frac{1}{R}\int_0^R (\widehat{\tau}_u\widehat{f})(1)\,\mathrm{d}u = \frac{1}{R}\int_0^R \varphi(\widehat{\tau}_u\widehat{f})\,\mathrm{d}u,$$

where φ is the observable $\varphi(\widehat{f}) \equiv \widehat{f}(1)$; this is in $L^1(\widehat{D}_{0>}, \widehat{\nu})$ since

$$\mathbb{E}(\varphi) = \int_{\widehat{D}_{0>}} \varphi(\widehat{f})\,\mathrm{d}\widehat{\nu}(\widehat{f}) = \int_{\widehat{D}_{0>}} \widehat{f}(1)\,\mathrm{d}\widehat{\nu}(\widehat{f}) = c_0.$$

Accordingly, by the Birkhoff Ergodic Theorem applied to $\widehat{\tau}_t$, we have (6.1) for $\widehat{\nu}$–a.e. $\widehat{f}$.

We shall prove that (6.1) also holds for $\overline{\nu}$–a.e. $\widehat{f}$. To this end, we claim that if (6.1) holds for some $\widehat{f}$, then it is true for all other points in the $\overline{\eta}_t$-orbit of $\widehat{f}$. Indeed, for fixed $s \in \mathbb{R}$, it is easily checked that $\widehat{f}(t)/(t-s)^\alpha$ and $\widehat{f}(t)/t^\alpha$ share the same log average, which is by hypothesis $c_0 \cdot \mathbb{I}(\Phi)$, hence equivalently $\overline{\eta}_s\widehat{f}(t)/t^\alpha$ and $\widehat{f}(t)/t^\alpha$.

Since the two flows $(\widehat{D}_{0>}, \widehat{\nu}, \widehat{\eta}_t)$, $(\widehat{D}_{0>}, \overline{\nu}, \overline{\eta}_t)$ share a common cross–section, this yields (6.1) for $\overline{\nu}$–a.e. $\widehat{f}$ and finishes the proof of Theorem 6.1. □

6.2. **The increment and renewal flows as stable horocycle flows.** We have seen in Proposition 4.2 that the scaling and increment flows obey the same commutation relation as the geodesic and stable horocycle flows g_t, h_t of a surface of constant negative curvature; here, making use of the results of the last section, weakening the definition of stable manifold allows us to make this analogy precise, whilst providing tools needed for the proof of Theorem 1.2.

Definition 6.1. *Given a flow g_t on a metric space (X,d), preserving a measure ν, we shall call a* stable horocycle flow *any flow h_t whose orbit of ν–a.e. $x \in X$ belongs to the g_t–stable manifold $W^s(x)$. We say h_t is a* Cesàro–average stable horocycle flow *if the orbit of ν–a.e. $x \in X$ belongs to $W^s_{CES}(x)$ (as defined in Theorem 1.1).*

Parts (i) and (iv) of the next proposition show the increment flow of the Mittag–Leffler process is a Cesàro–average horocycle flow for the scaling flow, while parts $(ii), (iv)$ show something similar for the renewal flow. Part (iii) is a related statement which we use in the proof of Theorem 1.2:

Proposition 6.1. *(Increment and renewal flows as Cesàro–average horocycle flows for $\widehat{\tau}_t$)*

(i) The orbit $\{\overline{\eta}_r\widehat{Z}\}_{r\in\mathbb{R}}$ is in $W^s_{CES}(\widehat{Z})$ as for $\widehat{\nu}$–a.e. path $\widehat{Z}$ then for any $r\in\mathbb{R}$ fixed,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T|\widehat{\tau}_t\widehat{Z}(1)-\widehat{\tau}_t(\overline{\eta}_r\widehat{Z})(1)|\,\mathrm{d}t=0.$$

(ii) With respect to the joining $\widehat{\mu}$ of Theorem 1.1, for a.e. pair $(\widehat{Z},\overline{N})$, the orbit $\{\overline{\eta}_r(h\circ\overline{N})\}_{r\in\mathbb{R}}$ is a subset of $W^s_{CES}(\widehat{Z})$ since for $\widehat{\mu}$–a.e. path $h\circ\overline{N}$, then for any $r\in\mathbb{R}$ fixed,

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T|\widehat{\tau}_t\widehat{Z}(1)-\widehat{\tau}_t(\overline{\eta}_r(h\circ\overline{N}))(1)|\,\mathrm{d}t=0.$$

(iii) Moreover, for $\overline{\mu}$–a.e. path $\overline{N}$, we have for each $r\in\mathbb{R}$ fixed:

$$\lim_{T\to\infty}\frac{1}{\log T}\int_1^T|\overline{\eta}_r(\overline{N})(t)-\overline{N}(t)|\,\frac{\mathrm{d}t}{\widehat{a}(t)\,t}=0.$$

(iv) All the statements remain true with $|\cdot|$ replaced by $||\cdot||^{\infty}_{[0,1]}$.

Proof. We know that $\Phi(\widehat{Z},t)\equiv\widehat{Z}(t)$ defines a cocycle over the increment flow $\overline{\eta}_t$. Now since $\widehat{Z}$ is nondecreasing, for fixed r and large t we have

$$|\widehat{\tau}_t(\overline{\eta}_r\widehat{Z})(1)-\widehat{\tau}_t\widehat{Z}(1)|=e^{-\alpha t}|\widehat{Z}(e^t+r)-\widehat{Z}(r)-\widehat{Z}(e^t)|\le e^{-\alpha t}(\widehat{Z}(e^t+r)-\widehat{Z}(e^t))+e^{-\alpha t}\,\widehat{Z}(r).$$

So by a logarithmic change of variables, proving (i) amounts to checking that for large T_0

$$\lim_{T\to\infty}\frac{1}{\log T}\int_{T_0}^T(\widehat{Z}(t+r)-\widehat{Z}(t))\,\frac{\mathrm{d}t}{t^{\alpha+1}}=0,\quad \widehat{\nu}-\text{a.s.}$$

Following the reasoning at the end of the proof of Theorem 6.1, this holds true.

For part (ii), as above, we have that for fixed r and for large t:

$$|(\overline{\eta}_r(h\circ\overline{N}))(t)-\widehat{Z}(t)|\le|h\circ\overline{N}(t+r)-\widehat{Z}(t+r)|+\widehat{Z}(t+r)-\widehat{Z}(t)+h\circ\overline{N}(r).$$

Using (1.7) and then following exactly the strategy used in the proof of (i), one shows (ii).

The proof of part (iii) follows the same pattern as that of (ii): from Corollary 1.1, the log average of $\overline{N}(t)/\widehat{a}(t)$ equals $\mathbb{E}(\widehat{Z}(1))$ and, by regular variation $\widehat{a}(t-r)\sim\widehat{a}(t)$ for t large.

Lastly, as in the proof of Theorem 1.1, these results also hold for $||\cdot||^{\infty}_{[0,1]}$: since $h(\cdot)$, $\overline{N}$ and $\widehat{Z}$ are nondecreasing, the same proofs go through. □

6.3. **Order–two ergodic theorems for the Mittag–Leffler and renewal flows and the renewal transformation: proofs of Theorem 1.2 and Corollary 1.2.**

Proof of Thm. 1.2. Part (i) follows directly from Thm. 6.1, since from Prop. 5.4 the two–sided Mittag–Leffler process is dual to an ergodic self–similar process with stationary ergodic increments.

For part (ii), part of the strategy for the proof of Thm. 1.2 still applies. First, by the Hopf theorem, it is sufficient to prove the statement for a specific cocycle. We choose $\Phi(\overline{N},t)=\overline{N}(t)$. For $\widehat{\mu}$−a.e. $\overline{N}$, statement (1.15) holds by Corollary 1.1. Part (iii) of Prop. 6.1 then shows the same statement is true for any path in the $\overline{\eta}_t$- orbit of $\overline{N}$. Now since the flows $\overline{\eta}_t$ and $\widehat{\eta}_t$ share a common cross–section, having the statement for a.e. point $\overline{N}$ with respect to the finite measure $\widehat{\mu}$ implies this for the infinite measure $\overline{\mu}$ as well. We are done with the proof of (1.15) and hence of Thm. 1.2. □

Next we move to discrete time and the:

Proof of Corollary 1.2. By the Markov property, the return times to the set A under the shift map are i.i.d. Therefore, for F the distribution function of the return times, N_n is a renewal process; indeed $N_n = \overline{N}(n)$ where $\overline{N}(t)$ is the continuous-time renewal process for the gap distribution μ_F.

We prove that (a) is equivalent to (b). That F is in the domain of attraction of G_α is equivalent to $1-F$ regularly varying of index $-\alpha$. This, in turn, is equivalent to saying that the renewal function $U(t) = \sum_{k\geq 0} F^{k\star}(t) = \mathbb{E}(\overline{N}(t)+1)$, and hence the return sequence $(\overline{a}_n)$, is regularly varying of index α; see p. 361 of [BGT87].

Proof of (i): Assuming (b), the function $\varphi = \chi_A$ generates a cocycle $\Psi(x,n)$ by equation (3.4). Applying (ii) of Theorem 1.2 to $\Psi(x,t)$, (i) of the corollary holds for $\Psi(x,n)$. By the Hopf Ratio Ergodic Theorem this then passes to any other cocycle over the transformation.

For an alternative argument, the process Y_i is a factor of the Markov shift, and is the event process model of a renewal transformation. The renewal shift model is a countable state Markov chain, so the argument just given applies to prove the order–two ergodic theorem for the renewal transformation. Now via the Ratio Ergodic Theorem this fact lifts to any conservative ergodic map which factors onto it, in particular, to the Markov shift. (More generally this passes on to any *similar* transformation; see Corollary 6 of [Fis92]).

The convergence of the Chung-Erdös averages of part (ii) are equivalent to the log averages of part (i) in this case, as mentioned in Proposition 1 of [ADF92].

We now prove (iii). Setting $\widehat{a}_n \equiv \frac{1}{1-F(n)}$, Feller [Fel49] (Thm. 7) proved that $N_n/\widehat{a}_n$ converges in law to a Mittag–Leffler distribution $\mathcal{M}_\alpha$. Again, from p. 361 of [BGT87], we know that $\overline{a}_n \sim \mathsf{c}\,\widehat{a}_n$, where $\mathsf{c} = (\Gamma(1-\alpha)\Gamma(1+\alpha))^{-1} = \mathbb{E}(\widehat{Z}(1))$; see (7.2) on p. 113 of [Fel49] for the value of $\mathbb{E}(\widehat{Z}(1))$.

Thus $N_n/\overline{a}_n$ converges to a rescaled Mittag–Leffler distribution of index α, of mean 1.

Now, F belongs to the domain of attraction of G_α so $S(n)/a(n) \xrightarrow{law} G_\alpha$, where $S(\cdot)$ is the polygonal interpolation extension of $(\overline{S}_n)$. Setting $N \equiv S^{-1}$ and $\widehat{a} \equiv a^{-1}$, $\forall x > 0$ we have $\mathbb{P}(S(n) \leq xa(n)) = \mathbb{P}(N(xa(n)) \geq n)$ and $\mathbb{P}(N(y)/\widehat{a}(y) \geq \widehat{a}(y/x)/\widehat{a}(y)) \to G_\alpha(x)$ as $y \to \infty$, with $y \equiv xa(n)$. As $\widehat{a}$ is regularly varying of index α, $\forall x \in (0,b], b > 0$, $\widehat{a}(y/x)/\widehat{a}(y)$ converges uniformly to $x^{-\alpha}$; see p. 22 of [BGT87]. So $N(n)/\widehat{a}(n) \xrightarrow{law} \mathcal{M}_\alpha$, thus $\widehat{a}(n) \sim \widehat{a}_n$ which yields $\overline{a}_n \sim \mathsf{c}\widehat{a}(n)$. □

Remark 6.1. Both the renewal process N_n and the cocycle $\Psi(x,n)$, which defines the increment shift model of the renewal transformation, $\widetilde{N}_n$, occured in the proof, but these are not quite the same: they differ by a time shift of the event process. Indeed, defining $\Psi^\circ(x,n) = \Psi(\sigma x, n)$, then Ψ° is also a cocycle over the renewal transformation, and from (1.4), $N_n = \Psi^\circ(x,n)$.

6.4. **Identification of the constant c: proof of Proposition 1.1.** We recall the notation regarding stable laws and processes from the beginning of §2. The law $G_\alpha = G_{\alpha,1}$ has an especially nice Laplace transform:

$$\mathbb{E}(e^{-wX}) = \exp(-\check{c}_\alpha\, w^\alpha), \qquad \check{c}_\alpha = \frac{\Gamma(3-\alpha)}{\alpha(1-\alpha)}. \tag{6.2}$$

Since the corresponding stable process Z has increasing paths with a dense set of jump points, then $\widehat{Z}$ has a nowhere dense set $C_{\widehat{Z}}$ of points of increase, with a flat stretch of $\widehat{Z}$ (on the gaps of $C_{\widehat{Z}}$) corresponding to each of the jumps of Z.

We recall the definition of Hausdorff φ–measure H_φ (see [Fal85], [Mat95]):

Definition 6.2. *Let $\varphi \in D_{0>}(\mathbb{R}^+)$, with d the Euclidean metric in $\mathbb{R}^n$. A δ-cover of $A \subseteq \mathbb{R}^n$ is a countable cover by subsets E_i of diameter $|E_i| < \delta$. Then*

$$H_\varphi(A) = \lim_{\delta \to 0} \left(\inf_{\{E_i\}:\, \delta-cover\, of\, A} \sum_{i=1}^{\infty} \varphi(|E_i|) \right).$$

We suppose the *gauge function* φ is *regularly varying of index α at zero*; that is, for some $\alpha > 0$, for each $a > 0$, $\varphi(at)/\varphi(t) \xrightarrow{t\to 0} a^\alpha$. The following scaling property, which expresses the self–similarity of the measure, follows from the regular variation of the gauge function, as noted in [BF92]:

Lemma 6.1. *If φ is regularly varying of index α at zero, then for every $a > 0$,*

$$H_\varphi(aA) = a^\alpha H_\varphi(A).$$

Definition 6.3. *Given a stochastic process $X(t)$ with paths in D, the* range *of X is the set–valued process given by taking the image of intervals, as follows:*

$$R_X(t) = \begin{cases} X([0,t]) & \text{for } t \geq 0 \\ X([t,0]) & \text{for } t < 0. \end{cases}$$

For $\alpha \in (0,1)$ we define

$$\underline{Z}(t) = \widehat{c}_\alpha^{1/\alpha} Z(t), \quad \text{where} \quad \widehat{c}_\alpha = 1/\check{c}_\alpha \text{ defined in (6.2)},$$

$$\psi(t) = t^\alpha \,(\log\log \frac{1}{t})^{1-\alpha}, \text{ and } \quad c_\alpha = \frac{\widehat{c}_\alpha}{\tilde{c}_\alpha} = \frac{\alpha^{1-\alpha}(1-\alpha)^\alpha}{\Gamma(3-\alpha)}, \text{ as in (1.12)}.$$

Lemma 6.2. (i) *(Hawkes) For ν–a.e. Z, for all $t \geq 0$,*

$$H_\psi(R_{\underline{Z}}(t)) = \tilde{c}_\alpha \cdot t, \qquad \textit{with} \quad \tilde{c}_\alpha = \alpha^\alpha (1-\alpha)^{1-\alpha}.$$

(ii) We have, for $\widehat{\nu}$–a.e. $\widehat{Z}$:

$$c_\alpha H_\psi(C_{\widehat{Z}} \cap [0,T]) = \widehat{Z}(T).$$

Proof. Using (6.2), $\underline{Z}$ satisfies

$$\mathbb{E}(e^{-\omega \underline{Z}(1)}) = \mathbb{E}(e^{-\omega\, \widehat{c}_\alpha^{1/\alpha} Z(1)}) = \exp(-\omega^\alpha\, \widehat{c}_\alpha\, \check{c}_\alpha) = e^{-\omega^\alpha},$$

and then part (i) is Theorem 2 of [Haw73].

To deduce (ii) from this, we denote by $\widehat{\underline{Z}}$ the generalized inverse of $\underline{Z}$ so $R_{\underline{Z}}(\widehat{\underline{Z}}(T)) = C_{\widehat{\underline{Z}}} \cap [0,T]$. On the other hand, as $\widehat{\underline{Z}}(t) = \widehat{Z}(t/\widehat{c}_\alpha^{1/\alpha})$, $C_{\widehat{\underline{Z}}} = \widehat{c}_\alpha^{1/\alpha}\, C_{\widehat{Z}}$.

We then set $t = \widehat{\underline{Z}}(T)$ and (i) says that

$$H_\psi(R_{\underline{Z}}(\widehat{\underline{Z}}(T)) = H_\psi(C_{\widehat{\underline{Z}}} \cap [0,T]) = H_\psi(\widehat{c}_\alpha^{1/\alpha}\,(C_{\widehat{Z}} \cap [0, T/\widehat{c}_\alpha^{1/\alpha}])) = \tilde{c}_\alpha\, \widehat{\underline{Z}}(T) = \tilde{c}_\alpha\, \widehat{Z}(T/\widehat{c}_\alpha^{1/\alpha}).$$

So $\forall T > 0$, since ψ is regularly varying of index α at zero, by Lemma 6.1 we prove (ii):

$$H_\psi(\widehat{c}_\alpha^{1/\alpha}\,(C_{\widehat{Z}} \cap [0,T])) = \widehat{c}_\alpha H_\psi(C_{\widehat{Z}} \cap [0,T]) = \tilde{c}_\alpha\, \underline{\widehat{Z}}(T).$$

□

We are now ready for the

Proof of Proposition 1.1. Since the scaling flow for $\widehat{Z}$ is ergodic, c_α times the right order–two density at $x = 0$ is, using (ii) and Birkhoff's ergodic theorem, and defining $f \in L^1(\widehat{D}_{0>}, \widehat{\nu})$ by $f(\widehat{Z}) = \widehat{Z}(1)$,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{c_\alpha H_\psi(C_{\widehat{Z}} \cap [x, e^{-s}])}{e^{-\alpha s}}\mathrm{d}s = \lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{\widehat{Z}(e^{-s})}{e^{-\alpha s}}\mathrm{d}s = \lim_{T\to\infty} \frac{1}{T}\int_0^T (\widehat{\tau}_{-s}\widehat{Z})(1)\mathrm{d}s =$$

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(\widehat{\tau}_{-s}\widehat{Z})\mathrm{d}s = \int_{\widehat{D}_{0>}} f\,\mathrm{d}\widehat{\nu} = \mathbb{E}(\widehat{Z}(1)) \equiv \mathsf{c} = \frac{1}{\Gamma(1-\alpha)\Gamma(1+\alpha)} = \frac{\sin\pi\alpha}{\pi\alpha},$$

see [BGT87], p.361. Thus for $\widehat{\nu}$–a.e. $\widehat{Z}$, the right order–two density at $x = 0$ exists and equals $c_\alpha^{-1}\mathsf{c}$. This is not enough: we want to show this holds at H_ψ–a.e. x in $C_{\widehat{Z}}$. The proof is a Fubini's theorem argument. Thus, writing $\widehat{D}_1 \subseteq D$ for the set of all $\widehat{Z}$ such that the right order–two density at 0 equals $c_\alpha^{-1}\mathsf{c}$, we let D_1 denote the corresponding set of paths $Z = \widehat{Z}^{-1}$. We have just seen that $\widehat{\nu}(\widehat{D}_1) = 1$, so equivalently $\nu(D_1) = 1$. Without loss of generality we can take D_1 to be Borel measurable, since it contains a Borel subset also of full measure (by Prop.28, Chapter 12 of [Roy68]).

From Proposition 4.1, we know η is jointly Borel measurable. Now considering, for $T > 0$,

$$A_T = \{(t, Z) \in [0, T] \times D : \eta_t Z \in D_1\};$$

this is a measurable set for the product of the Borel σ–algebras, hence is measurable in the $(m\times\nu)$-completion where m is Lebesgue measure. Since η_t preserves ν (Proposition 5.4), we know that for each $t \in [0, T]$, $\nu(\{Z : (t, Z) \in A_T\}) = \nu(\eta_t^{-1}(D_1)) = 1$. By Fubini's theorem therefore, for ν–a.e. Z, $m(\{t : (t, Z) \in A_T\}) = T$.

Now for ν–a.e. Z, m pushes forward by Z to the restriction of H_ψ to $C_{\widehat{Z}} \cap [0, T]$. Hence in conclusion for $\widehat{\nu}$–a.e. $\widehat{Z}$, we know that for H_ψ–a.e. x in $C_{\widehat{Z}} \cap [0, T]$ for any $T > 0$, the right order–two density at x exists and equals $c_\alpha^{-1}\mathsf{c}$. (Taking countable intersections over $T = 1, 2, \ldots$, we can replace $C_{\widehat{Z}} \cap [0, T]$ by $C_{\widehat{Z}}$). □

Remark 6.2. The constant c is also the order–two density at $+\infty$ (rather than at 0) both of $C_{\widehat{Z}}$ and of the integer fractal set $\mathcal{O}$ of renewal events. We can think of this second limit as defining a finitely additive Hausdorff measure: taking a long interval $[0, T]$, we cover the points in $\mathcal{O} \cap [0, T]$ with intervals of length 1 and then sum them up with regularly varying gauge function $\phi(r) = 1/\widehat{a}(1/r)$; the integer Hausdorff ϕ–measure of $\mathcal{O}$ is then equal to c.

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